WHEN IS A DERIVATOR REPRESENTABLE?

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Let $\mathbb{D}$ be a prederivator. The underlying diagram functors define a morphism of prederivators

$$\mathrm{dia}_{\mathbb{D}} : \mathbb{D} \to \mathcal{C}at(-, \mathbb{D}(e)).$$

A prederivator $\mathbb{D}$ is called *representable* if $\mathrm{dia}_{\mathbb{D}}$ is an equivalence of prederivators.

Let $\mathscr{C}$ be a right derivable category in the sense of [1]. The associated prederivator $\mathbb{D}(\mathscr{C})$ is a right derivator satisfying (Der5) [1, 2.21]. Suppose that the representable prederivator

$$X \mapsto \mathcal{C}at(X, \mathrm{Ho}(\mathscr{C}))$$

is also a right derivator and that the morphism $\mathrm{dia}_{\mathbb{D}(\mathscr{C})}$ is cocontinuous. Then we may regard the homotopy category $\mathrm{Ho}(\mathscr{C})$ as a right derivable category where every morphism is a cofibration and the weak equivalences are the isomorphisms. Moreover, the canonical functor

$$\gamma : \mathscr{C} \to \mathrm{Ho}(\mathscr{C})$$

is right exact in the sense of [1, 1.9]. Then dia is simply the morphism induced by γ. Since $\mathrm{dia}(e)$ is obviously an equivalence, [1, 3.20] implies that dia is also an equivalence of right derivators, i.e., $\mathbb{D}(\mathscr{C})$ is representable.

The fact that $\mathrm{Ho}(\mathscr{C})$ admits (co)limits is by itself insufficient for the argument. A counterexample is the derivator associated to the Waldhausen category of finitely generated stable modules over $\mathbb{Z}/p^2$. The homotopy category is equivalent to the category of finite dimensional $\mathbb{F}_p$-vector spaces (see [2, p. 1831]). Thus it is essential to know that dia is also cocontinuous (cf. [2, 4.5]).

On the other hand, assuming that dia is cocontinuous, it does not follow that γ is an equivalence. A trivial counterexample is the right derivable category of sets where every map is a weak equivalence. Its homotopy category is the terminal category.

It remains open whether $\mathbb{D}$ is always representable if $\mathrm{dia}_{\mathbb{D}}$ is cocontinuous. If this fails in general, it would show an interesting feature of derivators with (good) models. On the other hand, showing that this is true seems to require an extension of [1, 3.19,3.20] to abstract derivators.

References

[1] D.-C. Cisinski, *Catégories dérivables*, Bull. Soc. Math. France **138** (2010), no. 3, 317–393.

[2] F. Muro and G. Raptis, *A note on K-theory and triangulated derivators*, Adv. Math. **227** (2011), no. 5, 1827–1845.

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