Simplicial volumes, bounded cohomology, and Euler characteristics of (aspherical) manifolds

Workshop "Cobordisms, Strings, and Thom Spectra"
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Introduction

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Question (Gromov)

Let M be an oriented closed aspherical manifold. Does the following implication hold?

$$\|M\| = 0 \Longrightarrow \chi(M) = 0.$$

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- $(M, \partial M)$ oriented compact n-manifold

$$\|M, \partial M\| = \|[M, \partial M]\|_1 \geq 0.$$

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M hyperbolic n-manifold, then:
$$\|M\| = \frac{Vol(M)}{v_n} > 0. \quad \text{(Gromov-Thurston)}$$
(e.g. $\|\Sigma_g\| = 4g - 4$.)

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- **Comparison map**: $c_X^n\colon H_b^n(X;\mathbb{R}) \to H^n(X;\mathbb{R})$, $n \geq 0$.

Theorem (Duality principle)

M oriented closed connected n-manifold. Then:

1. $\|M\| = \| \; H_b^n(M;\mathbb{R}) \xrightarrow{c_M^n} H^n(M;\mathbb{R}) \overset{\cap[M]}{\cong} \mathbb{R} \; \|$.
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- (Löh-Moraschini-R.) If $\|M\| = 0$, then "at least half" of the cohomology classes of M are unbounded ($=$ not in the image of the comparison map).

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Let $f\colon X \to Y$ be a π_1-surjective map of path-connected spaces such that the kernel of $\pi_1(f)$ is amenable. Then: $H_b^\bullet(f;\mathbb{R})\colon H_b^\bullet(Y;\mathbb{R}) \xrightarrow{\cong} H_b^\bullet(X;\mathbb{R})$. So also: $H_b^\bullet(X;\mathbb{R}) \cong H_b^\bullet(B\pi_1(X);\mathbb{R})$ and $H_b^\bullet(BG;\mathbb{R}) \cong H_b^\bullet(*;\mathbb{R})$ if G is amenable.

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- These are the right derived functors of G-invariants in a category of Banach $\mathbb{R}[G]$-modules (Ivanov, Bühler,...). This is the functional analytic origin of bounded cohomology.

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Then the converse also holds (a relative version of Johnson's characterization of amenability):

Theorem (Moraschini-R.; converse to the Mapping Theorem)

Let $f\colon X \to Y$ be a π_1-surjective map of path-connected spaces with homotopy fiber F. Then: $f\colon X \to Y$ is amenable iff $\pi_1(F)$ is amenable.

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Theorem (Gromov's Vanishing Theorem)

Suppose that M admits an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ with the properties:

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A new proof approach : By (1), the Moore–Postnikov truncation of $U_\sigma \subseteq X$ (truncating π_1):

$$U_\sigma \to V_\sigma \to X$$

has the property that $H_b^\bullet(V_\sigma;\mathbb{R})$ is concentrated in degree 0. Note that U_σ may not be path-connected.

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A new proof approach : We obtain a diagram of cochain complexes:

$$\begin{array}{ccccc} C_b^\bullet(X;\mathbb{R}) & & \xrightarrow{c_X} & & C^\bullet(X;\mathbb{R}) \\ \downarrow & & & & \downarrow \\ C_b^\bullet(V_\sigma;\mathbb{R}) & \xrightarrow{c_{V_\sigma}} & C^\bullet(V_\sigma;\mathbb{R}) & \longrightarrow & C^\bullet(U_\sigma;\mathbb{R}) \end{array}$$

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A new proof approach : Taking homotopy limits and using excision:

$$\begin{array}{ccccc}
C_b^\bullet(X;\mathbb{R}) & & \xrightarrow{\quad c_X \quad} & & C^\bullet(X;\mathbb{R}) \\
\downarrow & & & & \downarrow \simeq \\
\mathrm{holim}_\sigma C_b^\bullet(V_\sigma;\mathbb{R}) & \longrightarrow & \mathrm{holim}_\sigma C^\bullet(V_\sigma;\mathbb{R}) & \longrightarrow & \mathrm{holim}_\sigma C^\bullet(U_\sigma;\mathbb{R})
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By (2), the homotopy limits are indexed by a poset of dimension $< n$. Hence, the bottom left cochain complex is concentrated in degrees $< n$. $\square$

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A new proof approach (R. '21): yields factorizations of the comparison map c_X of cochain complexes for general homotopy colimit decompositions of X,

$$\mathrm{hocolim}_I X_i \simeq X,$$

equipped with factorizations $(X_i \to Y_i \to X)$ through spaces Y_i with vanishing conditions on their bounded cohomology groups.

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- (**Generalized Milnor-Wood inequalities**) M occ smooth n-manifold and $(\pi\colon TM \to M)$ admits a flat structure. Then:

$$|\chi(M)| \leq \frac{\|M\|}{2^n} \text{ and the Euler class } e(M) \text{ is bounded.}$$

  (Ivanov-Turaev, Bucher-Monod, ...)

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  Remark: If M admits a flat metric, then: $\|M\| = 0 = \chi(M)$.

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- (**Boundedness of the Euler class**) M occ smooth n-manifold. *When is the Euler class* $e(M)$ *bounded?*

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- (**Boundedness of the Euler class**) M occ smooth n-manifold. *When is the Euler class $e(M)$ bounded?*

Proposition (Löh-Moraschini-R.)

(GromovQ) *is equivalent to:*

$e(M)$ is bounded.

Simplicial volume and the Euler characteristic (ctd.)

Recall (Gromov's question): M oriented closed aspherical n-manifold. Does the implication:

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- (**Amenability**) M occ n-manifold $n \geq 1$ with $\pi_1(M)$ amenable. Then:
  - (Gromov) $\|M\| = 0$.
  - (Sauer) If M is also aspherical, then $\chi(M) = 0$.

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Theorem (Löh-Moraschini-R.)

Let $n \in \mathbb{N}_{\geq 2}$ be even.

1. *There exist aspherical spaces X with an $H_*(-;\mathbb{Z})$-equivalence $X \to M$ to an occ n-manifold M such that $\|X\| = 0$ and $\chi(X) \neq 0$.*
2. *There exist occ n-manifolds M with an $H_*(-;\mathbb{Z})$-equivalence $X \to M$ from an aspherical space X such that $\|M\| = 0$ and $\chi(M) \neq 0$.*

Additivity of the simplicial volume

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- The Euler characteristic is the universal **additive** homotopy invariant. For compact manifolds $(M, \partial M)$ and $(N, \partial N)$ with $\partial M = \partial N$, we have:
$$\chi(M \cup_\partial N) = \chi(M) + \chi(N) - \chi(\partial M).$$

Additivity of the simplicial volume

- *What about the simplicial volume?*

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- There is additivity in a restricted sense: For oriented compact n-manifolds $(M, \partial M)$ and $(N, \partial N)$ with $\partial M = \partial N$ such that:

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- **Proposition.** (Löh-Moraschini-R.) (GromovQ) $\Longrightarrow$ (GromovQ$_\partial$).

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  - Objects: M oriented closed $(d-1)$-manifold (one from each diffeomorphism class)
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Remark: this property holds for products of ≥ 3 factors (Gromov). (But $\partial(M_1 \times M_2 \times M_3)$ is not aspherical, even if M_i and ∂M_i are aspherical and $\partial M_i \subset M_i$ are π_1-injective...)