What is a stable *n*-category?

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From (triangulated) homotopy categories to (stable) ∞-categories

Homotopy categories

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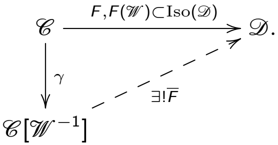

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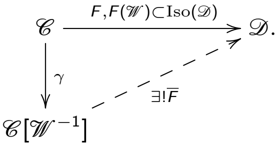

This is useful for defining **derived functors**.
Many invariants in homological algebra and homotopy theory are described as functors on the homotopy (or derived) category.

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- $\mathscr{C}[\mathscr{W}^{-1}]$ **lacks important category-theoretic properties.** *Many desirable constructions are not available in the context of homotopy categories (e.g. (homotopy) pushouts,* $\mathscr{C}^I[\mathscr{W}_I^{-1}] \neq \mathscr{C}[\mathscr{W}^{-1}]^I$*, etc.)*

Triangulated categories

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Triangulated structures axiomatize the structures often inherited by the homotopy category (e.g. $\mathbf{Ch(R)}[\mathscr{W}^{-1}]$) from specific properties of $(\mathscr{C}, \mathscr{W})$ and are used for studying (co)homological functors and other derived functors.

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A triangulated category consists of an additive category $\mathscr{D}$, an autoequivalence $\Sigma \colon \mathscr{D} \xrightarrow{\simeq} \mathscr{D}$ and a collection of *distinguished triangles* $X \to Y \to Z \to \Sigma X$ that satisfy the following properties:

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- $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is distinguished iff $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ is distinguished

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- for a commutative diagram between distinguished triangles

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- (the octahedral axiom) "$(Z/X)/(Y/X) \simeq Z/Y$"

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A stable ∞-category is an ∞-category $\mathscr{C}$ which has a zero object, admits finite (co)products, pushouts and pullbacks, and satisfies the property that a square

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Theorem (Lurie)

The homotopy category of a stable ∞-category admits a canonical triangulated structure.

From (stable) ∞-categories to homotopy n-categories

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Definition

A space X is n-truncated if $\pi_k(X,x) = 0$ for every $k > n$ and $x \in X$.

Example

- A set is 0-truncated (as a space). The circle S^1 is 1-truncated.
- Every space X has an n-truncation $X \to P_n(X)$ which kills the homotopy groups in degrees $> n$. (Similarly there are truncations of chain complexes.)

$(n,1)$-categories and higher homotopy categories (informally)

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Idea:

- categories enriched in $(n-1)$-truncated spaces $\rightsquigarrow$ $(n,1)$-categories;
- $(\infty,1)$-category $\mathscr{C}$ $\rightsquigarrow$ the **homotopy n-category** $\mathrm{h}_n\mathscr{C}$ is obtained by passing to the $(n-1)$-truncations of the mapping spaces.

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- An ∞-category $\mathscr{C}$ is a simplicial set such that every lifting problem

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- An ∞-category $\mathscr{C}$ is an n-category, $n \geq 1$, if:
  (i) given $f, f' \colon \Delta^n \to \mathscr{C}$ such that $f \simeq f'(\text{rel } \partial\Delta^n)$, then $f = f'$.
  ('equivalent n-morphisms are equal')
  (ii) given $f, f' \colon \Delta^m \to \mathscr{C}$, $m > n$, such that $f_{|\partial\Delta^m} = f'_{|\partial\Delta^m}$, then $f = f'$.
  ('no m-morphisms for $m > n$')

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Construction (Lurie): The set of m-simplices $\left(\mathrm{h}_n\mathscr{C}\right)_m$ of $\mathrm{h}_n\mathscr{C}$ is

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$\mathrm{h}_1\mathscr{C}$ is the usual homotopy category of $\mathscr{C}$.

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Definition

Let $\mathscr{C}$ be an ∞-category and $t \in \mathbb{Z}_{\geq -1}$.

- $x \in \mathscr{C}$ is weakly initial of order t if $\mathrm{map}_{\mathscr{C}}(x, y)$ is $(t-1)$-connected $\forall\ y \in \mathscr{C}$.

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- Let $F\colon K \to \mathscr{C}$ be a K-diagram in $\mathscr{C}$ where K is a simplicial set. A *weak colimit of F of order t* is a weakly initial object in $\mathscr{C}_{F/}$ of order t. ($\mathscr{C}_{F/}$ is the ∞-category of cocones on F.)

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Example

- $t = \infty$: standard notions of initial object and colimit in an ∞-category.
- $t = 0$: classical notions of weakly initial object and weak colimit.
- $t = -1$: any object/cone.

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Proposition

Let $\mathscr{C}$ be an ∞-category and $t \geq 1$. The full subcategory of weakly initial objects in $\mathscr{C}$ of order t is either empty or a t-connected ∞-groupoid. Therefore weakly initial objects of order t (if they exist) are unique up to (not necessarily unique) equivalence. (This fails for $t = -1, 0$.)

Example: Pushouts and higher weak pushouts

A square in an **ordinary category** $\mathscr{C}$

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$

is a **pushout** if for every $x \in \mathscr{C}$ the canonical map

$$\hom_{\mathscr{C}}(d, x) \to \hom_{\mathscr{C}}(b, x) \times_{\hom_{\mathscr{C}}(a,x)} \hom_{\mathscr{C}}(c, x)$$

is bijective.

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is a **weak pushout** (of order 0) if for every $x \in \mathscr{C}$ the canonical map

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is surjective (0-connected).

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Proposition (R.)

Let $\mathscr{C}$ be an ∞-category and $n \geq 1$.

- *Suppose that $\mathscr{C}$ has K-colimits where $\dim(K) = d$. Then $\mathrm{h}_n\mathscr{C}$ admits weak K-colimits of order $(n-d)$ and $\gamma_n \colon \mathscr{C} \to \mathrm{h}_n\mathscr{C}$ respects them.*
- *We have an equivalence of ∞-categories: $\mathrm{h}_{n-d}\big(\mathrm{h}_n(\mathscr{C}^K)\big) \simeq \mathrm{h}_{n-d}\big((\mathrm{h}_n\mathscr{C})^K\big)$.*
- *If $\mathscr{C}$ admits (finite) colimits, then $\mathrm{h}_n\mathscr{C}$ admits (finite) coproducts and weak pushouts of order $(n-1)$. If $\gamma_n \colon \mathscr{C} \to \mathrm{h}_n\mathscr{C}$ preserves finite colimits, then γ_n is an equivalence.*

Towards stable n-categories

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4. *For $n = 1$, this recovers the usual 2-category of triangulated categories.*

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The homotopy n-categories of stable ∞-categories should be the main examples of stable n-categories. **Question**: *Do (1)–(3) lead to a sufficiently good notion of stable n-category?*

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Theorem (R.)

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Problem: For $n = 1$

- this notion of stable 1-category is weaker than the notion of a triangulated category.

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- We could simply accept the case $n = 1$ as being a singular case at the lowest level of coherence. This would also justify the knotty aspects of triangulated category theory.
- Or we might prefer a notion of stable n-category with more structure (than properties) similar to triangulated structures (triangulated n-category?).

  This notion of triangulated n-category would be stronger in general than the notion of a stable n-category (except for the limiting case $n = \infty$).

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  This additional structure arises, for example, from the collection of (co)limits of K-diagrams in a stable ∞-category, for $\dim(K) \leq n$, and then passing to the homotopy n-category (similarly to the definition of distinguished triangles in the classical homotopy category).