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Universität Regensburg
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Seminar:
Katastrophentheorie – Catastrophe Theory
(Singularitäten differenzierbarer Abbildungen)
(WS 23/24, Mi 12-14, M103)

An Informal Overview of the Seminar

This is a short mathematical overview of the Seminar. The use of asterisks $(^*)$ *will indicate topics that will be discussed in more detail in the seminar talks.*

Let's consider a smooth (C^∞-)function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(\mathbf{0}) = 0$. Our main goal is to understand the behavior of f near the origin $\mathbf{0} \in \mathbb{R}^n$. For this reason, we focus on the **germ** $[f]$ **of** f, namely, the equivalence class of smooth functions which agree with f near the origin $(^*)$.

Can we classify all possible such germs? Since we are mainly interested in the qualitative and structural properties of the germ, we also introduce the following **equivalence relation**: we say that two germs $[f]$ and $[g]$ are **equivalent** if f and g agree near the origin up to a composition with a local diffeomorphism (= change of coordinates). Then the refined question becomes: what can we say about germs up to this notion of equivalence?

A distinguishing feature of a germ is the **type of singularity** that it defines at the origin. This singularity is also invariant under equivalences of germs. If the origin $\mathbf{0} \in \mathbb{R}^n$ is a **regular point of** f, then the germ of the function f is equivalent to the germ of the function

$$(x_1, \cdots, x_n) \mapsto x_1. \tag{1}$$

This is a consequence of the Implicit Function Theorem and a special case of the local description of submersions or smooth maps of constant rank $(^*)$.
On the other hand, if the origin is a **critical point of** f, *i.e.*, if all its first-order partial derivatives vanish at the origin, then there are two possibilities: either the singularity is **non-degenerate**, which means that the **Hessian matrix** $(^*)$

$$H(f) = (\frac{\partial^2 f}{\partial x_i \partial x_j})_{1 \leq i,j \leq n}$$

is non-singular at the origin, or othewise it is **degenerate**.

Example 1. Let's see what this means when $n = 1$. If $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ has a regular point at the origin, *i.e.*, $f'(0) \neq 0$, then up to a change of coordinates the function f agrees locally near the origin with the function $x \mapsto x$. If the point is critical but non-degenerate, *i.e.*, $f'(0) = 0$ and $f''(0) \neq 0$, then up to a change of coordinates the function f agrees locally near the origin with one of the functions $x \mapsto \pm x^2$.

This observation generalizes to the case of several variables. The germ of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ near a non-degenerate critical point is equivalent to the germ of a function of the form

$$\sum_{i=1}^{n} \pm x_i^2. \tag{2}$$

This is made precise by the Morse Lemma (*) which then gives a classification of germs in the case of non-degenerate critical points. The smooth functions whose critical points are non-degenerate are called **Morse functions**. Such functions have the following two important properties: **stability** and **genericity**. *Stability* means that every smooth function close enough to a Morse function is again a Morse function. *Genericity* means that every smooth function is arbitrarily close to a Morse function. The study of singularities of general types of smooth functions (or families of functions) in terms of their stability or genericity properties (locally or globally) is one of the main goals of the theory (and of singularity theory more generally).

Example 2. The example of the function $f(x) = x^3$ – which is not a Morse function – readily shows that small pertubations may change the nature of the singularities near the origin. For fixed $a \in \mathbb{R}$, the function $f_a(x) = x^3 + ax$ has two non-degenerate critical points for $a < 0$ and no critical points for $a > 0$! However, the function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (a, x) \mapsto f_a(x)$, regarded as a family of functions (**unfolding**) is stable in a suitable sense. This family is an example of a **universal unfolding** and it defines one of the **elementary catastrophes** (*).

What can we say about degenerate critical points in general? There is a generalization of the Morse Lemma which allows us to separate a germ into a quadratic (or Morse-like) part and a "totally non-quadratic" part. This is the **Reduction (or Splitting) Lemma** (*) which states that every germ can be described up equivalence by a function of the form

$$\pm x_1^2 \pm \cdots \pm x_r^2 + g(x_{r+1}, \cdots, x_n) \tag{3}$$

where r is the rank of the Hessian (note that $r < n$ if and only if the critical point is degenerate) and $g : \mathbb{R}^{n-r} \to \mathbb{R}$ is a "totally non-Morse" smooth function at the origin, *i.e.*, its Hessian matrix at the origin is zero. In this way we obtain a certain reduction of a germ to another germ of lower dimension but higher order of degeneracy.

The further development of the theory requires a careful study of the **algebra of germs** using their Taylor series expansions at the critical point (*). A **lemma of Borel** says that the algebra of germs surjects onto the algebra of formal power series (*).
Furthemore, a key question for the development of the theory is the following: what kind of qualitative information about the germ of f at $\mathbf{0}$ can we obtain from the part of the Taylor series expansion that consists only of partial derivatives of order $\leq k$? This polynomial part of the Taylor series of f at the origin is the **k-jet of** f at the origin (*).

Example 3. Let's see again what happens when $n = 1$. If $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ is such that $f'(0) = f^{(2)}(0) = \cdots = f^{(k-1)}(0) = 0$ and $f^{(k)}(0) \neq 0$, then up to a change of coordinates f agrees locally near the origin with (one of) the function(s) $x \mapsto \pm x^k$ (*).

If not all first-order partial derivatives of f vanish at the origin, then the origin is a regular point and we know from (1) what the germ of f looks like up to equivalence. In this case, we say that the germ is **1-determined**, *i.e.*, it is determined by its 1-jet up to equivalence. If all first-order partial derivatives of f vanish at the origin, but the second-order derivatives are so that the Hessian matrix is non-singular, then the origin is a non-degenerate critical point, and again we know from (2) what the germ of f looks like up to equivalence – thus, it is **2-determined**. When is the germ of f determined by its k-jet for some finite k? The question leads to the study of necessary and sufficient conditions for the **(finite) determinacy of germs** (*). Note that, using the Reduction Lemma, we may restrict our attention to the case of a totally degenerate singularity (= the Hessian matrix vanishes).

Finite determinacy of germs is closely related to an (algebraically defined) invariant, the **codimension of a germ** (*). It can be shown that the codimension is finite if and only if the germ is finitely determined (*).

One of the main results will be **Thom's classification theorem** for germs of functions at (totally) degenerate critical points of codimension ≤ 4 (*): there are essentially seven types of such singulatiries and these give rise to the **seven elementary catastrophes**.

A key notion in the theory is that of a **(universal) unfolding** of f (*). This essentially sees f as an element of a family of functions which now depends on additional parameters (see Example 2 above for an example of an unfolding). The notion of an unfolding leads to the study of families of functions as a way of better understanding the possible **metamorphoses of singularity loci**. In the case of a universal unfolding, the family should be large enough to contain all possible local evolutions of the original germ (*).

The theorem of the **existence and uniqueness of universal unfoldings** for finitely determined germs is one of the main results in the theory and its proof is based on the notion of **transversality** (*).

Although degenerate germs are not stable (nor generic), it turns out that their **universal unfoldings are stable** (and generic). So we have exchanged the stability (and genericity) of functions for the stability of those unfoldings of functions which contain all possible local manifestations of instability so that they become stable as unfoldings (*).

A **catastrophe** refers exactly to a point where the nature of the singularities for a family of functions changes suddenly (*i.e.*, a singularity in the locus of singularities of an unfolding) – see Example 2. There are seven elementary catastrophes which correspond to the seven types of singulaties of totally degenerate germs of codimension ≤ 4 (*). These have been used for the qualitative analysis of a variety of phenomena that involve qualitative changes of form.

Recommended Reading for the Seminar

[Br] Th. Bröcker, *Differentiable germs and catastrophes.* Translated from the German, last chapter and bibliography by L. Lander. London Mathematical Society Lecture Note Series No. 17. Cambridge University Press, Cambridge-New York-Melbourne, 1975.

[CH] Domenico P. L. Castrigiano; Sandra A. Hayes, *Catastrophe theory.* Second edition. With a foreword by René Thom. Westview Press, Advanced Book Program, Boulder, CO, 2004.

[Lu] Yung-Chen Lu, *Singularity theory and an introduction to catastrophe theory.* Universitext, Springer-Verlag, New York-Berlin, 1980.

[PS] Tim Poston; Ian Stewart, *Catastrophe theory and its applications.* With an appendix by D. R. Olsen, S. R. Carter, and A. Rockwood. Surveys and Reference Works in Mathematics No. 2. Pitman, London-San Francisco, Calif.-Melbourne, 1978.

[Br] and [CH] are the main references for the Seminar. Both contain a careful and detailed treatment of the theory with complete proofs. We will mainly follow the exposition of [CH]. [Lu] (Chapters 1–4) contains a very readable introduction to the theory, with an emphasis on motivation and applications, and a readable overview of the results. [PS] is recommended for a less technical introduction to the theory which focuses on the intuitive geometric meaning of the main results and their applications.