A PROOF OF THE GILLET-WALDHAUSEN THEOREM

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Let $\mathcal{E}$ be an exact category which is closed under kernels of surjections in an ambient abelian category $\mathcal{A}$ (see [4]). Let $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})$ denote the category of bounded chain complexes in $\mathcal{E}$ where the cofibrations are the degreewise admissible monomorphisms and the weak equivalences are the quasi-isomorphisms (as chain complexes in $\mathcal{A}$). We let $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})_{[a,b]}$ denote the full Waldhausen subcategory of those chain complexes which are concentrated in degrees $[a,b]$, $a \leq b$. Our goal for the proof of the Gillet-Waldhausen theorem is to determine the K-theory of this Waldhausen subcategory for all $a \leq b$.

We consider a larger class of weak equivalences in $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})_{[a,b]}$ which consists of the chain maps which induce isomorphisms on H_i for $i < b$. This class of weak equivalences defines a new Waldhausen structure on $\mathrm{Ch}^{\mathrm{b}}{}_{[a,b]}(\mathcal{E})$ (with the same cofibrations), which we denote by $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})^{<b}_{[a,b]}$. Moreover, this new Waldhausen category is saturated (i.e. it has the "2-out-of-3" property) and admits (functorial) factorizations (i.e. every morphism can be written as the composition of a cofibration followed by a weak equivalence). The factorizations are given by the standard mapping cylinder construction truncated in degrees $> b$. Note that the suspension functor on $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})^{<b}_{[a,b]}$, iterated $b-a+1$ times, is weakly trivial. As a consequence, the K-theory of the Waldhausen category $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})^{<b}_{[a,b]}$ is homotopically trivial.

Let $\mathscr{E}$ be the full Waldhausen subcategory of $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})_{[a,b]}$ of those chain complexes which have trivial homology in degrees $< b$. In other words, these are exactly the chain complexes which become weakly trivial in $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})^{<b}_{[a,b]}$. Applying the Fibration Theorem[1], we obtain a homotopy fiber sequence

$$K(\mathscr{E}) \longrightarrow K(\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})_{[a,b]}) \longrightarrow K(\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})^{<b}_{[a,b]}) \simeq *$$

[1]Waldhausen's Fibration Theorem in [3, 1.6.4] (see also [1, A.3]) requires also the extension axiom, which $\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})^{<b}_{[a,b]}$ does not satisfy. However, a small modification of the proof in [3] shows that this axiom is not needed. We recall that the proof in [3, 1.6.4] uses the extension axiom in order to identify $v\overline{w}\mathcal{S}_\bullet\mathcal{C}$ with $v\mathcal{S}_\bullet F_\bullet(\mathcal{C}, \mathcal{C}^w)$. But the inclusion $v\overline{w}\mathcal{S}_\bullet\mathcal{C} \subset v\mathcal{S}_\bullet F_\bullet(\mathcal{C}, \mathcal{C}^w)$ is always a weak equivalence because we have weak equivalences for each $n \geq 0$,

$$\begin{array}{ccc} & v\mathcal{S}_\bullet\mathcal{C} \times v\mathcal{S}_\bullet\mathcal{S}_n\mathcal{C}^w & \\ {\scriptstyle\simeq}\swarrow & & \searrow{\scriptstyle\simeq} \\ v\mathcal{S}_\bullet\overline{w}_n\mathcal{C} = v\overline{w}_n\mathcal{S}_\bullet\mathcal{C} & \longrightarrow & v\mathcal{S}_\bullet F_n(\mathcal{C}, \mathcal{C}^w) \end{array}$$

by the Additivity Theorem. Here $\overline{w}_n\mathcal{C}$ denotes the Waldhausen subcategory of diagrams in $\mathcal{C}$

$$c_0 \overset{\sim}{\rightarrowtail} c_1 \overset{\sim}{\rightarrowtail} \cdots \overset{\sim}{\rightarrowtail} c_n$$

with the usual cofibrations and where the (v-)weak equivalences are defined pointwise. This has Waldhausen subcategories which are identified with $\mathcal{C}$, embedded as constant diagrams, and $\mathcal{S}_n\mathcal{C}^w$, embedded as diagrams with $c_0 = *$, and there is an equivalence of categories between $\overline{w}_n\mathcal{C}$ and $E(\mathcal{C}, \overline{w}_n\mathcal{C}, \mathcal{S}_n\mathcal{C}^w)$.

and therefore the map $K(\mathscr{E}) \longrightarrow K(\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})_{[a,b]})$ is a homotopy equivalence.

There are exact functors (of Waldhausen categories):

(i) the b-th homology functor $H_b \colon \mathscr{E} \to \mathcal{E}$. This is well-defined because $\mathcal{E} \subset \mathcal{A}$ is closed under kernels of surjections in $\mathcal{A}$.

(ii) the inclusion $i \colon \mathcal{E} \to \mathscr{E}$ of chain complexes concentrated in degree b.

The composite $H_b \circ i$ is the identity functor and the composite $i \circ H_b$ is weakly equivalent to the identity functor. Hence $K(i) \colon K(\mathcal{E}) \simeq K(\mathscr{E}) \colon K(H_b)$. As a consequence, the inclusion $j_b \colon \mathcal{E} \to \mathrm{Ch}^{\mathrm{b}}(\mathcal{E})_{[a,b]}$ of chain complexes concentrated in degree b induces a homotopy equivalence

$$K(j_b) \colon \mathcal{E} \xrightarrow{\simeq} \mathrm{Ch}^{\mathrm{b}}(\mathcal{E})_{[a,b]}. \tag{1}$$

For $a \leq k \leq b$, we consider the inclusion $j_k \colon \mathcal{E} \to \mathrm{Ch}^{\mathrm{b}}(\mathcal{E})_{[a,b]}$ of chain complexes concentrated in degree k. By the Additivity Theorem, for each $a \leq k < b$, the induced map $K(j_k)$ agrees up to sign with the map $K(j_{k+1})$. It follows inductively that $K(j_k)$ is a homotopy equivalence for each $a \leq k \leq b$. In particular, the inclusion functor, for $n \geq 0$,

$$j_0 \colon \mathcal{E} \to \mathrm{Ch}^{\mathrm{b}}(\mathcal{E})_{[-n,n]},$$

induces a homotopy equivalence

$$K(j_0) \colon \mathcal{E} \xrightarrow{\simeq} \mathrm{Ch}^{\mathrm{b}}(\mathcal{E})_{[-n,n]}. \tag{2}$$

Passing to the colimit of the homotopy equivalences (2) as $n \to \infty$, we obtain the homotopy equivalence of the Gillet-Waldhausen theorem [2, 1.11.7], [4, V.2.2]:

$$K(\mathcal{E}) \xrightarrow{\simeq} K(\mathrm{Ch}^{\mathrm{b}}(\mathcal{E})). \tag{3}$$

References

[1] Marco Schlichting, *Negative K-theory of derived categories*, Math. Z. 253 (2006), no. 1, 97-134.

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