Higher homotopy categories and Brown representability

Algebra Seminar
Masaryk University, Brno

George Raptis
University of Regensburg

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AFTs and Brown representability

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Let $\mathscr{C}$ be a (locally small) category with small coproducts, weak pushouts, and a set of compact weak generators (e.g. $\mathrm{Ho}(CW_*^c)$, $\mathrm{Ho}(\mathrm{spectra})$, etc.).

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Question 2:
Is there a general context for Brown representability which unifies classical Brown representability and LAFT for space-valued presheaves?

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Definition

We say that an ∞-category $\mathscr{C}$ has an *essentially small colimit-dense subcategory* if there is an essentially small full subcategory $\mathscr{C}_0 \subset \mathscr{C}$ such that every object $x \in \mathscr{C}$ is a colimit of a diagram $K \to \mathscr{C}_0 \subset \mathscr{C}$ with values in $\mathscr{C}_0$ for some small simplicial set K.

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Let $\mathscr{C}$ be a locally small cocomplete ∞-category which has an essentially small colimit-dense subcategory and let $\mathscr{D}$ be a locally small ∞-category. Then a functor $F\colon \mathscr{C} \to \mathscr{D}$ is a left adjoint if and only if F preserves small colimits.

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- F is a left adjoint if and only if $F_{/d}$ admits a terminal object for all $d \in \mathscr{D}$.

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- Show that $F_{/d}$ has a weakly terminal object.

□

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The goal is to introduce a Brown representability context which:

- generalizes from ordinary categories to $(n, 1)$-categories
- generalizes from classical homotopy categories to higher homotopy categories
- uses suitable generalizations of the notions of *weak colimit*, *weak generator*, and *compact object*
- bridges the gap between classical Brown representability and LAFT.

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Proposition

Let $\mathscr{C}$ be an ∞-category. Then: $\mathscr{C} \simeq$ (n-category) if and only if $\mathrm{map}_{\mathscr{C}}(x, y)$ is $(n-1)$-truncated for all $x, y \in \mathscr{C}$.

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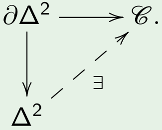

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2. $t = 0$: classical notions of weakly initial object and weak colimit.

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Let $\mathscr{C}$ be an ∞-category and K a simplicial set. The comparison between colimits in $\mathscr{C}$ and in $\mathrm{h}_n\mathscr{C}$ is related to the properties of the canonical forgetful functors

$$\Phi_n^K \colon \mathrm{h}_n(\mathscr{C}^K) \to \mathrm{h}_n(\mathscr{C})^K.$$

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4. *Suppose that $\mathscr{C}$ admits finite colimits. If $\gamma_n \colon \mathscr{C} \to \mathrm{h}_n\mathscr{C}$ preserves finite colimits, then γ_n is an equivalence.*

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- $n = \infty$: stable ∞-category.
- $n = 1$: weaker than triangulated category. (a singular case?)

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Remark (The role of n)

- *A weakly cocomplete ∞-category is a cocomplete ∞-category.*
- *A cocomplete n-category is weakly k-cocomplete for all $k \geq n$.*
- *An n-category is weakly $(n+1)$-cocomplete if and only if it is cocomplete.*
- *A weakly cocomplete n-category $\mathscr{C}$ admits weak K-colimits of order $(n - \dim(K))$.*

Compactly generated *n*-categories

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Definition

Let $\mathscr{C}$ be a locally small ∞-category.

- A set of *weak generators* of $\mathscr{C}$ is a set of objects $\mathscr{G}$ that jointly detect equivalences: a morphism $f : x \to y$ in $\mathscr{C}$ is an equivalence if and only if $\mathrm{map}_{\mathscr{C}}(g, x) \to \mathrm{map}_{\mathscr{C}}(g, y)$ is an equivalence for every $g \in \mathscr{G}$.

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- Suppose that every diagram $T\colon \mathbb{N} \to \mathscr{C}$ is equipped with a *distinguished* cone $T^{\triangleright}\colon \mathbb{N}^{\triangleright} \to \mathscr{C}$ with cone object $\mathrm{colim}^{w} T$.
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A locally small n-category $\mathscr{C}$ is called *compactly generated* if $\mathscr{C}$ is a weakly cocomplete n-category and has a set of weak generators $\mathscr{G}$ which consists of compact objects.

Compactly generated *n*-categories (ctd.)

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Let $\mathscr{C}$ be a locally small weakly cocomplete n-category. We say that $\mathscr{C}$ *satisfies Brown representability* if for any given functor $F\colon \mathscr{C}^{\mathrm{op}} \to \mathcal{S}_{<n}$, F is representable if (and only if) the following conditions are satisfied.

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- F sends small coproducts in $\mathscr{C}$ to products in $\mathcal{S}_{<n}$.
- For every weak pushout in $\mathscr{C}$ of order $(n-1)$

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ z & \longrightarrow & w \end{array}$$

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As a consequence, every localization of a compactly generated n-category also satisfies Brown representability.

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  A functor $F\colon \mathscr{C} \to \mathscr{D}$ is a left adjoint if and only if F preserves small coproducts and sends pushouts to <u>weak</u> pushouts.

Higher Brown representability (ctd.)

Corollary

Let $\mathscr{C}$ be a presentable ∞-category and let $\mathscr{D}$ be a locally small n-category.

1. *Suppose that $\mathscr{C}$ is a stable. Then $\mathrm{h}_n\mathscr{C}$ satisfies Brown representability.* *As a consequence, a functor $F\colon \mathrm{h}_n\mathscr{C} \to \mathscr{D}$ is a left adjoint if and only if F preserves small coproducts and weak pushouts of order $(n-1)$.*
2. *Suppose that $n \geq 2$. Then $\mathrm{h}_n\mathscr{C}$ satisfies Brown representability.* *As a consequence, a functor $F\colon \mathrm{h}_n\mathscr{C} \to \mathscr{D}$ is a left adjoint if and only if F preserves small coproducts and weak pushouts of order $(n-1)$.*

Remark

- *$n = 1$: classical Brown representability context (Heller); $n = \infty$: context of LAFT*
- *Every locally presentable category $\mathscr{C}$ satisfies Brown representability.* *A functor $F\colon \mathscr{C} \to \mathscr{D}$ is a left adjoint if and only if F preserves small coproducts and sends pushouts to weak pushouts.*
- *(Heller) $\mathrm{h}_1(\mathscr{S})$ does not satisfy Brown representability.*

General AFTs (without compact generators)

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Question 3: What about AFTs for general weakly (co)complete *n*-categories?

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Let $\mathscr{C}$ and $\mathscr{D}$ be ∞-categories and let $F \colon \mathscr{C} \to \mathscr{D}$ be a functor.

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Theorem ("n-GAFTs"; Nguyen-R.-Schrade)

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- *Suppose that $\mathscr{C}$ is a finitely weakly cocomplete n-category. Then F is a left adjoint if and only if F preserves finite coproducts, weak pushouts of order $(n-1)$, and satisfies the h-terminal object condition.*

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- *Suppose that $n \geq 3$, $\mathscr{C}$ is a locally small weakly cocomplete n-category and $\mathscr{D}$ is 2-locally small.*

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