Higher homotopy categories and K-theory

Algebraic Topology Seminar
University of Warwick

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Today: How much of $K(\mathscr{C})$ can be recovered from $\mathrm{h}_n(\mathscr{C})$ (or from $\mathbb{D}^{(n)}_{\mathscr{C}}$)?

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Proposition

Let $\mathscr{C}$ be an ∞-category. Then:

$$\mathscr{C} \simeq (n-\textit{category}) \iff \mathrm{map}_{\mathscr{C}}(x,y) \textit{ is } (n-1)-\textit{truncated}\ \forall x, y \in \mathscr{C}.$$

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Construction. (Lurie) The set of *m*-simplices $\big(\mathrm{h}_n\mathscr{C}\big)_m$ of $\mathrm{h}_n\mathscr{C}$ is

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The 0-simplices (objects) of $\mathrm{h}_1\mathscr{C}$ are the objects of $\mathscr{C}$. The 1-simplices (morphisms) are the morphisms of $\mathscr{C}$ up to homotopy. The 2-simplices (composition) correspond to equivalence classes of diagrams:

$$\begin{tikzcd} \partial\Delta^2 \arrow[r] \arrow[d] & \mathscr{C}. \\ \Delta^2 \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

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1. ($t = \infty$): We recover the standard notions of initial object and colimit in an ∞-category.
2. ($t = 0$): We recover the classical notions of weakly initial object and weak colimit.

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Let $\mathscr{C}$ be an ∞-category and K a simplicial set. The comparison between colimits in $\mathscr{C}$ and in $\mathrm{h}_n\mathscr{C}$ is related to the properties of the canonical forgetful functors

$$\Phi_n^K \colon \mathrm{h}_n(\mathscr{C}^K) \to \mathrm{h}_n(\mathscr{C})^K.$$

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Fact. The inclusion $\Omega|\mathrm{Ob}\ \mathsf{S}_\bullet^{\simeq}\mathscr{C}| \xrightarrow{\simeq} K(\mathscr{C})$ is a homotopy equivalence.

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Let $\mathscr{C}$ be a pointed ∞-category with finite colimits and $n \geq 1$. We equip $\mathrm{h}_n\mathscr{C}$ with the **canonical structure** can, which consists of those squares in $\mathrm{h}_n\mathscr{C}$

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Suppose that $\mathscr{C}$ is a stable ∞-category.

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Corollary (conjectured by Antieau, connective case)

Let $\mathscr{C}$ and $\mathscr{C}'$ be stable ∞-categories such that $(\mathrm{h}_n\mathscr{C}, \mathrm{can}) \simeq (\mathrm{h}_n\mathscr{C}', \mathrm{can})$. Then the $(n-1)$-truncations of K-theory are equivalent: $P_{n-1}K(\mathscr{C}) \simeq P_{n-1}K(\mathscr{C}')$.

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This is called the n-derivator associated to $\mathscr{C}$ and retains many of the structural properties of $\mathscr{C}$.

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Remark. The map $K(\mathscr{C}) \to K(\mathbb{D}^{(1)}_{\mathscr{C}})$ is not a π_3-iso in general. (Muro-R.)