

Higher homotopy categories and K -theory

Algebraic Topology Seminar
University of Warwick

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23 June 2020

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Today: How much of $K(\mathcal{C})$ can be recovered from $h_n(\mathcal{C})$ (or from $\mathbb{D}_{\mathcal{C}}^{(n)}$)?

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Proposition

Let \mathcal{C} be an ∞ -category. Then:

$$\mathcal{C} \simeq (n\text{-category}) \iff \text{map}_{\mathcal{C}}(x, y) \text{ is } (n-1)\text{-truncated } \forall x, y \in \mathcal{C}.$$

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Construction. (Lurie) The set of m -simplices $(h_n\mathcal{C})_m$ of $h_n\mathcal{C}$ is

$$\frac{\{\text{sk}_n\Delta^m \rightarrow \mathcal{C} \text{ which extend to } \text{sk}_{n+1}\Delta^m\}}{\simeq \text{ relative to } \text{sk}_{n-1}\Delta^m}$$

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The 0-simplices (objects) of $h_1\mathcal{C}$ are the objects of \mathcal{C} . The 1-simplices (morphisms) are the morphisms of \mathcal{C} up to homotopy. The 2-simplices (composition) correspond to equivalence classes of diagrams:

$$\begin{array}{ccc} \partial\Delta^2 & \xrightarrow{\quad} & \mathcal{C}. \\ \downarrow & \nearrow \exists & \\ \Delta^2 & & \end{array}$$

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- 2 ($t = 0$): We recover the classical notions of weakly initial object and weak colimit.

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Proposition

Let \mathcal{C} be an ∞ -category, $n \geq 1$.

- 1 Suppose that \mathcal{C} has K -colimits where $\dim(K) = d$. Then $\mathbf{h}_n\mathcal{C}$ admits weak K -colimits of order $(n - d)$ and $\gamma_n: \mathcal{C} \rightarrow \mathbf{h}_n\mathcal{C}$ preserves these.

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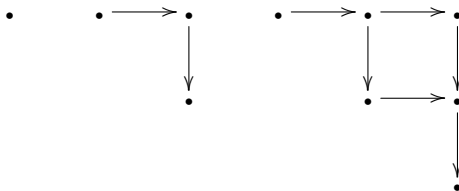
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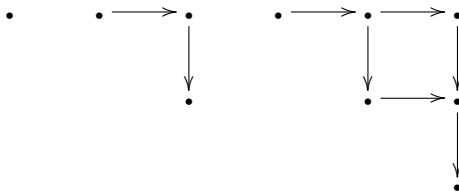
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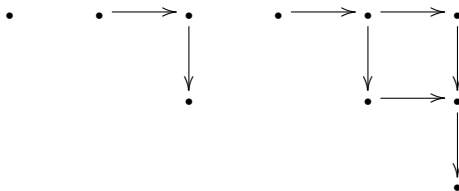
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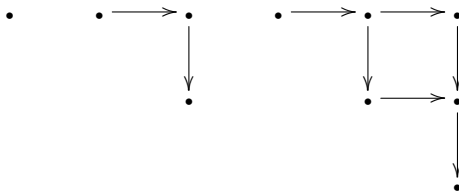


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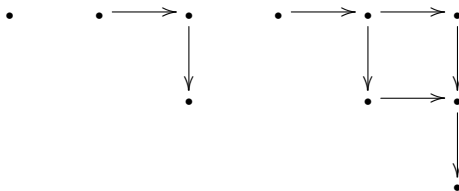


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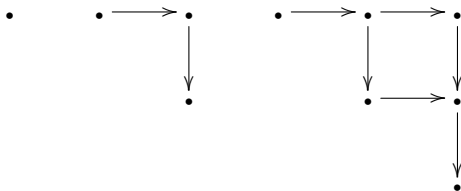
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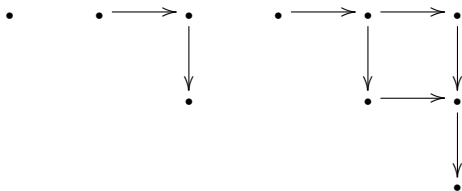
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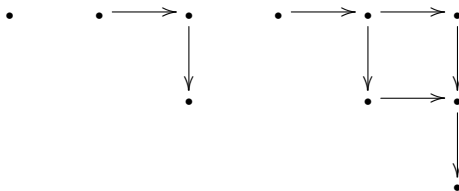
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Fact. The inclusion $\Omega|\mathrm{Ob} S_{\bullet}^{\simeq}\mathcal{C}| \xrightarrow{\sim} K(\mathcal{C})$ is a homotopy equivalence.

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which are equivalent to a square that arises from a pushout square in \mathcal{C} .
(This is additional structure only for $n = 1$!)

We may use these squares to define an analogous S_\bullet -construction for $\mathbf{h}_n \mathcal{C}$.
(All we need is a structure of *distinguished squares*.)

Let $S_n(\mathbf{h}_n \mathcal{C}, \text{can}) \subseteq \text{Fun}(\text{Ar}[n], \mathbf{h}_n \mathcal{C})$ be the full subcategory spanned by $X: \text{Ar}[n] \rightarrow \mathbf{h}_n \mathcal{C}$ such that:

- all diagonal values of X are zero objects.
- every square in the diagram X is in can .

Let $S_n^\simeq(\mathbf{h}_n \mathcal{C}, \text{can}) \subset S_n(\mathbf{h}_n \mathcal{C}, \text{can})$ be the maximal ∞ -groupoid in $S_n(\mathbf{h}_n \mathcal{C}, \text{can})$.
Then $[n] \mapsto S_n^\simeq(\mathbf{h}_n \mathcal{C}, \text{can})$ is a simplicial space.

Definition

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K -theory of higher homotopy categories II

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Let \mathcal{C} be a pointed ∞ -category with finite colimits. Then the canonical comparison map $K(\mathcal{C}) \rightarrow K(\mathbf{h}_n \mathcal{C}, \text{can})$ is n -connected.

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Suppose that \mathcal{C} is a stable ∞ -category.

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Corollary (conjectured by Antieau, connective case)

Let \mathcal{C} and \mathcal{C}' be stable ∞ -categories such that $(\mathrm{h}_n\mathcal{C}, \mathrm{can}) \simeq (\mathrm{h}_n\mathcal{C}', \mathrm{can})$. Then the $(n - 1)$ -truncations of K -theory are equivalent: $P_{n-1}K(\mathcal{C}) \simeq P_{n-1}K(\mathcal{C}')$.

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Remark. The map $K(\mathcal{C}) \rightarrow K(\mathbb{D}_{\mathcal{C}}^{(1)})$ is not a π_3 -iso in general. (Muro-R.)