Titles and abstracts of the workshop "Recent advances in bounded cohomology" funded by SFB 1085 *Higher Invariants*

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Invited talks:

Quasi-morphisms on Surface Diffeomorphism Groups

Jonathan Bowden

University of Regensburg

We discuss the problem of constructing quasi-morphisms on the group of diffeomorphisms of a surface that are isotopic to the identity, thereby resolving a problem of Burago-Ivanov-Polterovich from 2006. This is achieved by considering a new kind of curve graph, in analogy to the (by now) classical curve graph first studied by Harvey in the 70's, on which the full diffeomorphism group naturally act by isometries. Joint with S. Hensel and R. Webb.

Invariant cocycles on the Furstenberg boundary

Michelle Bucher

University of Geneva

Recently, Nicolas Monod showed that the evaluation map $H^*_m(G \curvearrowright G/P) \longrightarrow H^*_m(G)$ between the measurable cohomology of the action of G on its Furstenberg boundary G/P and the measurable cohomology of G is sujective with a kernel that can be entirely described in terms of invariants in the cohomology of the maximal split torus $A < G$. I will describe explicitly such cocycles on G/P in low degree for G a product of isometries of real hyperbolic spaces and $G = \mathrm{SL}(3, \mathbb{C})$. As a consequence, we will see that the comparison map $H^*_{m,b}(G, \mathbb{R}) \rightarrow H^*_m(G, \mathbb{R})$ from the measurable bounded cohomology is injective in degree 3, which is new for nontrivial products of isometries of hyperbolic spaces. Joint work with Alessio Savini.

An ℓ^1-norm inequality for complete manifolds

Caterina Campagnolo

Autonomous University of Madrid

In the 80's, Gromov proved his "Main inequality", relating the simplicial volume of a manifold to its volume, under some curvature assumptions.
Since then the community has tried to generalize and enhance this relation by weakening the curvature assumptions, extending, or improving the inequality.
In joint work with Shi Wang, we extend the results of Besson-Courtois-Gallot about the ℓ^1-norm of the fundamental class of a closed manifold to all homology classes of a complete manifold. Our inequalities are sharper than Gromov's original ones and are expressed in terms of the critical exponent of the manifold.

Bounded cohomology over non-Archimedean fields

Francesco Fournier-Facio

ETH Zürich

When we work in bounded cohomology, it usually goes without saying that we work over the reals. But what happens if instead the field we work over is non-Archimedean, such as the p-adics? In this case, the topology and the algebra interact in a very nice way, which allows to prove extremely strong results. I will give an overview of what some of the most surprising ones are, and explain why they hold over non-Archimedean fields and not over the reals. We all know that bounded cohomology (over the reals) is hard to compute, but this should serve as a reminder of why this is the case.

Efficient cycles for cusped hyperbolic manifolds

Roberto Frigerio

University of Pisa

Let M be a connected orientable manifold. An efficient cycle for M is the limit (in an appropriate measure space) of a sequence of fundamental cycles whose ℓ^1-norm converges to the simplicial volume of M. Computing the simplicial volume is usually a hard task, and exhibiting an efficient cycle can be even harder. In the case of finite volume hyerpbolic manifolds, a celebrated result by Gromov and Thurston exhibits an explicit efficient cycle, and Jungreis and Kuessner later proved that Gromov-Thurston's smearing construction actually provides the unique efficient cycle for a huge class of finite volume hyperbolic manifolds, including all the closed ones. Here we prove that a finite-volume hyperbolic manifold admits more than one efficient cycle if and only if it is commensurable with the figure-8 knot complement.
This is a joint work with Ennio Grammatica and Bruno Martelli.

Uniform homotopy theory

Sanjeevi Krishnan

The Ohio State University

The goal of this talk is to describe a uniform-theoretic refinement of an equivalence between classical homotopy theories of cubical sets and topological spaces salient to bounded cohomology. Concretely, geometric realization lifts to a full and faithful embedding from a Lipschitz homotopy category of cubical sets into a uniform homotopy category of uniform spaces, including metric spaces. Cubical and singular variants of bounded cohomology, at least when restricted to cubical sets with extra structure on the cubes and path-connected spaces, generalize to representable functors on these different homotopy categories which coincide where appropriate. This talk describes the representing objects in detail, discusses how other variants of bounded cohomology (e.g. polynomially bounded cohomology) are interpretable in this framework, and discusses the (potential) role of uniform homotopy colimits in bounded cohomology.
Joint work with Crichton Ogle.

Barycenter method in bounded cohomology

Jean-François Lafont

The Ohio State University

The barycenter map technique was pioneered by Besson-Courtois-Gallot in the early 1990s, and provides a powerful tool for establishing rigidity theorems. I'll explain how the technique can be used to analyze the comparison map from bounded cohomology to ordinary cohomology, and will survey some of the results obtained via this technique.

The L^1-metric on $Diff_0(M, area)$, braids and quasimorphisms

Michał Marcinkowski

University of Wrocław

Let M be a compact Riemannian manifold. There are a number of interesting metrics on the group of volume preserving diffeomorphisms of M, among them the L^1-metric. If M is an $(n > 2)$-dimensional disc, then the diameter of $Diff_0(M, vol)$ with L^1-metric is finite by the celebrated result of A. Shnirelman. In the 2-dimensional case the situation is very different. In this talk I will present recent results on the geometry of the L^1-metric on $Diff_0(M, area)$ where M is a compact surface. In particular, that there are many L^1-Lipschitz quasimorphismsm on $Diff_0(M, area)$ and that all right-angled Artin groups embed quasi-isometrically into $Diff_0(M, area)$. Joint work with M. Brandenbursky and E. Shelukhin.

Asymptotic cohomology and Ulam stability

Nicolas Monod

EPFL

The goal of this talk is to advertise a sort of "non-standard bounded cohomology" that we are developing in joint work with Lev Glebsky, Alex Lubotzky and Bharatram Rangarajan.
This theory is at the same time very similar to bounded cohomology, but also has some mysterious aspects of its own.
Our main motivation is to apply it to prove that some groups enjoy the "stability of unitary representation", which is a non-commutative generalization of the property that quasimorphisms are close to homomorphisms.

Bounded and unbounded cohomology of diffeomorphism groups

Sam Nariman

Purdue University

The plane bundles over surfaces are classified by their Euler class and Milnor showed that if they admit flat structures, the Euler class should be a bounded class. This result has been generalized in many directions and in particular for a connected finite-dimensional Lie group G, it has been extensively studied the conditions under which the characteristic classes of flat principal G-bundles are bounded classes viewed in the cohomology of the classifying space of the group G with the discrete topology. In this talk, I will talk about a joint work with N. Monod, in which we started studying the same question for diffeomorphism groups in different regularities that are infinite-dimensional topological groups. In particular, we computed the bounded cohomology of $\mathrm{Diff}_0(S^1)$ and $\mathrm{Diff}_0(D^2)$ entirely and showed that they are isomorphic to the polynomial ring generated by the Euler class. We also answered Ghys' question about generalizing Milnor-Wood inequality to flat S^3-bundles by showing that the Euler class is unbounded in $H^4(\mathrm{Diff}_0(S^3); \mathbb{R})$.

Low-degree cohomology, higher property (T) phenomena and stability of arithmetic groups

Roman Sauer

Karlsruhe Institute of Technology

We show that that the cohomology of an arithmetic lattice in a semisimple Lie group G is completely determined by G below the rank of G. This holds true for a class of representations that includes all unitary representations.
Unlike the work of Borel, Wallach or later Franke on the cohomology of arithmetic groups, our approach does not use global analysis on the symmetric space in a direct way. It relies on properties of the coarse geometry of the lattice. One specific tool is cohomology with

polynomial growth constraints. This is joint work with Uri Bader. The cohomological result is an important ingredient in joint work with Uri Bader, Alex Lubotzky and Shmuel Weinberger on the stability (and non-stability) of arithmetic groups which we'll talk about briefly as well.

Contributed talks:

The simplicial volume of contractible 3-manifolds

Giuseppe Bargagnati

University of Pisa

We show that the simplicial volume of a contractible open 3-manifold not homeomorphic to $\mathbb{R}^3$ is infinite. To this aim, we show that the excision of a compact connected codimension-0 submanifold with amenable and π_1-injective boundary induces isometric isomorphisms in ℓ^1-homology in degree at least 2. With similar techniques, we compute the spectrum of simplicial volume of irreducible open 3-manifolds. This is a joint work with my advisor Roberto Frigerio.

Filling volumes of homotopy equivalences

Federica Bertolotti

Scuola Normale Superiore of Pisa

The simplicial volume of mapping tori is a well studied topic and yet many open questions remain in high dimension: for example, in dimension 5 the only existing result states that there exists a mapping torus with positive simplicial volume, while in even dimension greater than 4 there are no existing results at all.
The integral and the real filling volume are numerical invariants defined on orientation preserving self-homotopy equivalences of closed oriented manifolds. T hey can be defined naively as kinds of stable filling norms for maps. The interest in these topological invariants arises from the fact that they give an alternative way to estimate the simplicial volume and the stable integral simplicial volume of a mapping torus starting from its monodromy and a fundamental cycle of its fiber.
During this talk I will introduce these two invariants, explaining their relations with the simplicial volume and the stable integral simplicial volume of mapping tori. Time permitting, some applications will be given.

From classical homological stability to stability for bounded cohomology

Thorben Kastenholz

University of Göttingen

Classically, homological stability refers to the phenomenon that the homology of a nested sequence of groups agrees in more and more degrees.
The most common proof strategy is due to Quillen and relies on a spectral sequence argument and sufficiently nice simplicial complexes on which the groups in question act. It was shown

in a paper by Hartnick and De la Cruz how the spectral sequence argument can be adapted to bounded cohomology. In this talk I will explain how to investigate the classically occuring simplicial complexes from the perspective of bounded cohomology and illustrate that in many cases classical homological stability also yields stability in bounded cohomology. All of this is joint work with Robin Sroka.

Amenable covers and vanishing theorems

Kevin Li

University of Southampton

Gromov and Ivanov proved: If a space X can be covered by n subsets with amenable fundamental group, then the comparison map vanishes in all degrees $\geqslant n$. Similarly, Sauer showed: If additionally X is aspherical, then the ℓ^2-Betti numbers and the homology growth vanish in all degrees $\geqslant n$.
I will present a unified setup for proving these vanishing results using classifying spaces for families of subgroup. I will also mention a vanishing theorem for the relative simplicial volume of manifolds with π_1-injective boundary. This is joint work with Clara Löh and Marco Moraschini.

Bounded cohomology via differential forms and cup product

Domenico Marasco

University of Pisa

Integrating over straight simplices defines a map from the space of closed differential k-forms of a negatively curved Riemannian manifold to its degree k bounded cohomology. In particular, in a 1988 paper J.Barge and E.Ghys showed that the case of closed surfaces S and $k = 2$ is particularly interesting since this map is injective and thus $\Omega^2(S)$ defines an infinite dimensional subspace of $H_b^2(S)$. We will have a look at some facts about bounded cohomology classes defined by differential forms. Then we will show that the cup product of a class defined by an exact 2-form with any other class is always trivial in bounded cohomology.

Weakly bounded, but unbounded, 2-classes in finitely presented groups

Francesco Milizia

Scuola Normale Superiore of Pisa

I will talk about a recent work in which Dario Ascari and I exhibit a finitely presented group with the following property: it has a second cohomology class which is not bounded, but is "weakly bounded" according to a definition introduced by Neumann and Reeves while they were studying coarse-geometric aspects of central extensions. This gives counterexamples to a conjecture of Gromov about bounded primitives of differential 2-forms on universal covers of closed manifolds.

Implementing Amenable Groups in Lean

Matthias Uschold

University of Regensburg

Proof assistants permit to formally verify whether a mathematical proof is correct, i.e. if the statement can be deduced from the hypotheses using a fixed set of logical rules and axioms.
Recently, the Lean Theorem Prover has gained a lot of attention, mainly thanks to the mathlib, a library containing proofs of many theorems covered in undergraduate courses.
In this talk, I will explain why it is desirable for researchers to make use of proof assistants, and I will demonstrate how proving theorems in Lean feels like, following an attempt of implementing amenable groups.