

SOME CHARACTERIZATIONS OF ACYCLIC MAPS

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ABSTRACT. We discuss two categorical characterizations of the class of acyclic maps between spaces. The first one is in terms of the higher categorical notion of an epimorphism. The second one employs the notion of a balanced map, that is, a map whose homotopy pullbacks along π_0 -surjective maps define also homotopy pushouts. We also identify the modality in the homotopy theory of spaces that is defined by the class of acyclic maps, and discuss the content of the generalized Blakers-Massey theorem for this modality.

1. INTRODUCTION AND STATEMENT OF RESULTS

One of the most enigmatic elementary constructions in homotopy theory is Quillen's plus construction. For each based path-connected space X there is a based acyclic map $q_X: X \rightarrow X^+$ such that the kernel of $\pi_1(q_X)$ is exactly the maximal perfect subgroup of $\pi_1(X)$. Such a construction first appeared for homology spheres in [7] and it was later developed by Quillen [10] who used it for the purpose of defining higher algebraic K -theory of rings. The construction easily generalizes as follows: for each perfect normal subgroup $P \trianglelefteq \pi_1(X)$, there is an acyclic map $q_{X,P}: X \rightarrow X_P^+$ such that the kernel of $\pi_1(q_{X,P})$ is P . The map $q_{X,P}$ is universal with respect to this property and therefore also unique up to homotopy.

We recall that a map of path-connected spaces $f: X \rightarrow Y$ is *acyclic* if its homotopy fiber F_f is acyclic, i.e., $\tilde{H}_*(F_f; \mathbb{Z}) = 0$. Acyclic maps are also characterized by the following properties (see [4, 5]):

- (H) the map f induces isomorphisms $H_*(X; f^*A) \cong H_*(Y; A)$ for all local abelian coefficient systems A on Y .
- (+) the map f is identified up to weak homotopy equivalence with the plus construction $q_{X,P}: X \rightarrow X_P^+$ with respect to the kernel P of $\pi_1(f, x)$ – which is a perfect group.

More generally, we say that a map $f: X \rightarrow Y$ of (not necessarily path-connected) spaces is *acyclic* if the homotopy fibers of f are acyclic spaces (equivalently, if (H) is satisfied). An acyclic map is therefore a homology equivalence in a strong sense, but it can be very far from inducing a π_* -isomorphism in general. For example, the theorem of Kan and Thurston [6] shows that for any path-connected space X , there is a discrete group G and an acyclic map $BG \rightarrow X$. Note that acyclic maps are closed under homotopy pullbacks (by definition) and under homotopy pushouts (by (H)). Property (+) connects the acyclic maps with the plus construction and therefore with algebraic K -theory. We refer to [4] for a nice treatment of the approach to algebraic K -theory based on the plus construction.

This paper makes no special claim to originality. Its purpose is twofold, firstly, to prove two further characterizations of the class of acyclic maps, and secondly, to discuss the properties of the modality in the homotopy theory of spaces that is

defined by the class of acyclic maps. The first characterization involves the notion of an epimorphism as suggested by higher category theory. In the homotopy theory (∞ -category) of spaces, this notion corresponds to the following: a map $f: X \rightarrow Y$ is called a (*homotopy*) *epimorphism* if the commutative square

$$(1) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ f \downarrow & & \downarrow \text{id} \\ Y & \xrightarrow{\text{id}} & Y \end{array}$$

is a homotopy pushout. This is the categorical dual of the notion of monomorphism as defined in higher category theory. (See [8] for a discussion of this notion in general ∞ -categories.) The first characterization states that a map is acyclic if and only if it is an epimorphism.

The second characterization is motivated by another well-known special property of acyclic maps. Given an acyclic map between path-connected spaces $f: X \rightarrow Y$, then the homotopy fiber sequence

$$(2) \quad F_f \rightarrow X \xrightarrow{f} Y$$

is also a homotopy cofiber sequence (see [5, 1]). Indeed, the induced map from the homotopy cofiber $C_{F_f \rightarrow X} \rightarrow Y$ is acyclic (by excision) and induces a π_1 -isomorphism (by the van Kampen theorem). However, it is easy to see that this property cannot characterize the acyclic maps (the trivial map $X \rightarrow *$ always has this property, too). We consider the following stronger property: a map $f: X \rightarrow Y$ is called *balanced* if for every 0-connected map $g: B \rightarrow Y$, the homotopy pullback square

$$(3) \quad \begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow f \\ B & \xrightarrow{g} & Y \end{array}$$

is also a homotopy pushout. We note that 0-connected maps, that is, maps which are surjective on π_0 , are the *effective epimorphisms* in the homotopy theory of spaces (in the sense of [8]). The second characterization states that a map is acyclic if and only if it is balanced. This characterization follows easily from the results of Alonso in [1] – where general criteria for a homotopy pullback of (path-connected) spaces to be a homotopy pushout are shown. We prove these two characterizations of acyclic maps, as epimorphisms and as balanced maps, in Section 2.

The notion of a *modality* was used by Anel-Biedermann-Finster-Joyal [2] in connection with generalizations of the classical Blakers-Massey theorem. The prototypical example of a modality is the factorization system in the ∞ -topos of spaces which is defined by the classes of n -connected and n -truncated maps. It is shown in [2] that there is a Blakers-Massey theorem and a dual Blakers-Massey theorem associated to each modality. These theorems specialize to the classical statements in the case of the aforementioned modality in spaces.

In Section 3, we give a proof that the class of acyclic maps defines the left class of a modality in spaces and identify the associated right class as the class of maps $f: X \rightarrow Y$ such that $\ker(\pi_1(f, x))$ does not contain a non-trivial perfect subgroup

for any $x \in X$. Then we apply the main results of [2] and comment on the content of the corresponding Blakers-Massey theorems for this example of a modality.

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2. EPIMORPHISMS, ACYCLIC MAPS, AND BALANCED MAPS

In this section we prove two characterizations of acyclic maps, as epimorphisms in the homotopy theory of spaces, and as balanced maps. The characterization of acyclic maps as balanced maps is essentially contained in the work of Alonso [1], for path-connected spaces at least, and our proof is closely related to arguments used in that paper (see [1, Theorem 2.5 and Corollary 2.9]). See also [1, Proposition 4.2] for some further characterizations of acyclic maps.

Theorem 2.1. *Let $f: X \rightarrow Y$ be a map of spaces.*

- (a) *f is acyclic if and only if f is an epimorphism.*
- (b) *f is acyclic if and only if f is balanced.*

Proof. (a). Suppose that f is an epimorphism. By the Mayer-Vietoris sequence of the homotopy pushout square (1), it follows that f satisfies condition (H), and therefore it is acyclic.

Suppose that f is acyclic. By restricting to path-connected components if necessary, we may assume that both X and Y are (non-empty and) path-connected. Let C be the homotopy pushout:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ f \downarrow & & \downarrow \\ Y & \xrightarrow{c} & C. \end{array}$$

By excision, the map c induces homology isomorphisms for all local abelian coefficient systems on C , and therefore c is acyclic by condition (H). It is also a π_1 -isomorphism, by the van Kampen theorem, since $\pi_1(f, x)$, $x \in X$, is surjective (and therefore an epimorphism in the category of groups). It follows that c is a weak homotopy equivalence.

(b). Suppose that f is acyclic. Let $g: B \rightarrow Y$ be a 0-connected map and consider the homotopy pullback

$$(4) \quad \begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow f \\ B & \xrightarrow{g} & Y. \end{array}$$

We claim that this square is also a homotopy pushout. Since (4) is obtained as the coproduct of the homotopy pullbacks over each path component of Y , and also coproducts of homotopy pushouts are again homotopy pushouts, we may assume that Y is (non-empty and) path-connected. (These facts are instances of homotopy

descent in an ∞ -topos.) Then X is path-connected, too. We first consider the case where B is also path-connected. We have a diagram of homotopy pullbacks:

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & X \\ \downarrow & & \downarrow f' & & \downarrow f \\ * & \longrightarrow & B & \xrightarrow{g} & Y \end{array}$$

where F is the homotopy fiber of both f and f' . Since f and f' are acyclic, the left and the composite squares are also homotopy pushouts - see the remarks after (2) above. Therefore the right square is also a homotopy pushout. This shows the claim in the case where B is path-connected.

Let $B = \bigsqcup_I B_i$ where each B_i is a path-connected component of $B \neq \emptyset$ (we may assume that B is a CW complex). Then each homotopy pullback

$$\begin{array}{ccc} E_i & \xrightarrow{g'_i} & X \\ \downarrow & & \downarrow f \\ B_i & \xrightarrow{g_i} & Y \end{array}$$

is also a homotopy pushout since f is acyclic and B_i is path-connected. Consider the diagram

$$\begin{array}{ccccc} \bigsqcup_I E_i & \xrightarrow{\bigsqcup_I g'_i} & \bigsqcup_I X & \xrightarrow{\nabla} & X \\ \downarrow & & \downarrow \bigsqcup_I f & & \downarrow f \\ B = \bigsqcup_I B_i & \xrightarrow{\bigsqcup_I g_i} & \bigsqcup_I Y & \xrightarrow{\nabla} & Y. \end{array}$$

By the definition of homotopy pullbacks, the composite square is a homotopy pullback. (This is an instance of homotopy descent in an ∞ -topos [8].)

The left square is a homotopy pushout because it is so for each $i \in I$. Moreover, the right square is a homotopy pushout using the fact that f is acyclic (or an epimorphism): the map f is up to homotopy the composition of taking $|I| (> 0)$ iterated homotopy pushouts along copies of the map f . Therefore the composite square is also a homotopy pushout and the claim in the general case follows.

Suppose that f is balanced. The claim is trivial if Y is empty, so we assume that $Y \neq \emptyset$. Given a homotopy pullback

$$(5) \quad \begin{array}{ccc} E & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ B & \xrightarrow{g} & Y \end{array}$$

consider the square of homotopy fibers of the maps which start from each corner of the square (5) and map to Y , taken over a point $y \in Y$,

$$(6) \quad \begin{array}{ccc} F_{gf',y} & \longrightarrow & F_{f,y} \\ \downarrow & & \downarrow \\ F_{g,y} & \longrightarrow & * \end{array}$$

This square is again a homotopy pullback. Therefore $F_{gf',y} \simeq F_{g,y} \times F_{f,y}$.

Assuming that g is 0-connected, the first square (5) is a homotopy pushout since f is balanced. It follows by well-known properties of the homotopy theory of spaces that the square of homotopy fibers (6) is also a homotopy pushout. (This is an instance of homotopy descent in an ∞ -topos [8].) Thus, the homotopy pushout $F_{g,y} * F_{f,y}$ of (6) is weakly contractible for any $g: B \rightarrow Y$ which is π_0 -surjective. Setting, for example, $g: B = Y \times S^0 \rightarrow Y$ to be the projection, it follows that $F_{f,y}$ must have trivial reduced homology for every $y \in Y$. This completes the proof of Theorem 2.1. \square

Remark 2.2. (On the connectivity assumption.) The assumption that the map $g: B \rightarrow Y$ is 0-connected in the definition of a balanced map is necessary. Without this assumption, Theorem 2.1(b) fails for obvious reasons (consider, for example, the case where $B = \emptyset$). More specifically, without this assumption on g , the resulting notion simply defines the class of weak homotopy equivalences.

Remark 2.3. The ∞ -category of spaces is well-copowered since quotient objects (in the sense of the notion of epimorphism used here) correspond to perfect normal subgroups of the fundamental group (at different basepoints).

Example 2.4. Applying Theorem 2.1(a) to the map $f: X \rightarrow *$, we recover as a special case the following well-known characterization: X is acyclic if and only if ΣX is weakly contractible.

Remark 2.5. An acyclic map $f: X \rightarrow Y$ does *not* have the property that each homotopy pushout square

$$\begin{array}{ccc} X & \longrightarrow & C \\ f \downarrow & & \downarrow \\ Y & \longrightarrow & D \end{array}$$

is also a homotopy pullback. For example, given an acyclic space F , the homotopy pushout

$$\begin{array}{ccc} F & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma F \simeq * \end{array}$$

is not a homotopy pullback. We will return to this dual question in the next section.

Example 2.6. (Algebraic K -theory.) Let R be a unital ring and $BGL(R) \rightarrow BGL(R)^+$ the acyclic map to the path-connected cover of the algebraic K -theory of R . As a consequence of Theorem 2.1(b), for each map $\emptyset \neq B \rightarrow BGL(R)^+$ the homotopy pullback

$$\begin{array}{ccc} E & \longrightarrow & BGL(R) \\ \downarrow & & \downarrow \\ B & \longrightarrow & BGL(R)^+ \end{array}$$

is a homotopy pushout. For $B = *$, a model for the homotopy fiber E is given by the (acyclic) *Volodin space*. When $B = S^n$, then the homotopy pullback E satisfies $E^+ \simeq S^n$.

Example 2.7. (Using the Kan-Thurston theorem.) Let X be a path-connected space with universal covering space $p: \tilde{X} \rightarrow X$. According to the Kan-Thurston theorem [6], there is a discrete group G and an acyclic (based) map $f: BG \rightarrow X$. We have a homotopy pullback:

$$(7) \quad \begin{array}{ccc} BP & \longrightarrow & BG \\ \downarrow & & \downarrow f \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

where P is the kernel of $\pi_1(f)$ – a perfect normal subgroup of G . Then, by Theorem 2.1(b), the square (7) is also a homotopy pushout.

3. THE MODALITY OF ACYCLIC MAPS

3.1. Preliminaries. For the presentation of the results in this section, it will be convenient to use the language of ∞ -categories as in [2]. Let \mathcal{S} denote the ∞ -category of spaces. Given $i: A \rightarrow B$ and $p: X \rightarrow Y$ in an ∞ -category \mathcal{C} , we say that i and p are *orthogonal* if the following square

$$\begin{array}{ccc} \mathrm{map}_{\mathcal{C}}(B, X) & \longrightarrow & \mathrm{map}_{\mathcal{C}}(B, Y) \\ \downarrow & & \downarrow \\ \mathrm{map}_{\mathcal{C}}(A, X) & \longrightarrow & \mathrm{map}_{\mathcal{C}}(A, Y) \end{array}$$

is a pullback in \mathcal{S} . In this case, we say that i is *left orthogonal* to p , or p is *right orthogonal* to i , and denote this relation by $i \perp p$. For a class of morphisms A in an ∞ -category \mathcal{C} , we write

$$\begin{aligned} A^\perp &:= \{p \in \mathcal{C} \mid a \perp p \text{ for every } a \in A\} \\ {}^\perp A &:= \{i \in \mathcal{C} \mid i \perp a \text{ for every } a \in A\}. \end{aligned}$$

Definition 3.1. Let \mathcal{C} be an ∞ -category. A *factorization system* in \mathcal{C} consists of a pair of classes of morphisms $(\mathcal{L}, \mathcal{R})$ in \mathcal{C} such that:

- (a) every morphism f in \mathcal{C} admits a factorization $f = \mathcal{R}(f) \circ \mathcal{L}(f)$ where $\mathcal{L}(f) \in \mathcal{L}$ and $\mathcal{R}(f) \in \mathcal{R}$.
- (b) $\mathcal{L}^\perp = \mathcal{R}$ and $\mathcal{L} = {}^\perp \mathcal{R}$.

We say that \mathcal{L} is the *left class* and \mathcal{R} is the *right class* of the factorization system.

Definition 3.2. Let \mathcal{C} be an ∞ -topos. A factorization system $(\mathcal{L}, \mathcal{R})$ in an ∞ -topos \mathcal{C} is a *modality* if the class \mathcal{L} is closed under pullbacks.

We refer to [2, Section 3] for a discussion of the general properties of factorization systems and modalities.

3.2. Acyclic maps define a modality. Our purpose in this section is to give a proof that the class of acyclic maps in the ∞ -topos \mathcal{S} is the left class of a modality. Let \mathcal{A} denote the class of acyclic maps in \mathcal{S} . We recall that $f: X \rightarrow Y$ is acyclic if its homotopy fibers are acyclic spaces (equivalently, f induces homology isomorphisms for all local abelian coefficient systems on Y). Furthermore, let \mathcal{B} denote the class of maps $f: X \rightarrow Y$ in \mathcal{S} such that for each $x \in X$ ($= x: 1 \rightarrow X$), the normal subgroup

$$\ker(\pi_1(f, x)) \trianglelefteq \pi_1(X, x)$$

does *not* contain a non-trivial perfect subgroup. Equivalently, $f \in \mathcal{B}$ if and only if for each $x \in X$ the (homotopy) fiber $F_{f,y}$ of f at $y = f(x)$ has the property that its fundamental group $\pi_1(F_{f,y}, x)$ has trivial maximal perfect subgroup. (This equivalence uses the fact that the epimorphism $\pi_1(F_{f,y}, x) \twoheadrightarrow \ker(\pi_1(f, x))$ defines a central extension, and therefore it preserves the maximal perfect subgroup by [4, 1.6].)

Theorem 3.3. *The pair $(\mathcal{A}, \mathcal{B})$ is a modality in \mathcal{S} .*

We will need the following lemma. We first recall that a map $f: X \rightarrow Y$ in \mathcal{S} is a *monomorphism* if it is equivalent to an inclusion of components (equivalently, if the canonical map $\Delta f: X \rightarrow X \times_Y X$ is an equivalence).

Lemma 3.4. *Let X be a 0-connected space, $P \trianglelefteq \pi_1(X, x)$ a perfect normal subgroup, and let $q_{X,P}: X \rightarrow X_P^+$ denote the associated plus construction. Then the restriction map*

$$q_{X,P}^*: \text{map}_{\mathcal{S}}(X_P^+, Z) \rightarrow \text{map}_{\mathcal{S}}(X, Z)$$

is a monomorphism in \mathcal{S} for every $Z \in \mathcal{S}$. Moreover,

$$q_{X,P}^*: \pi_0(\text{map}_{\mathcal{S}}(X_P^+, Z)) \hookrightarrow \pi_0(\text{map}_{\mathcal{S}}(X, Z))$$

is identified with the inclusion of the classes of maps $g: X \rightarrow Z$ for which $\ker(\pi_1(g, x))$ contains the subgroup P .

Proof. Since $X \rightarrow X_P^+$ is an epimorphism in \mathcal{S} , by Theorem 2.1(a), it follows that the induced square in \mathcal{S}

$$\begin{array}{ccc} \text{map}_{\mathcal{S}}(X_P^+, Z) & \longrightarrow & \text{map}_{\mathcal{S}}(X_P^+, Z) \\ \downarrow & & \downarrow \\ \text{map}_{\mathcal{S}}(X_P^+, Z) & \longrightarrow & \text{map}_{\mathcal{S}}(X, Z) \end{array}$$

is a pullback. Therefore the first claim follows. The second claim is a well-known consequence of the universal property of the plus construction (see, for example, [4, Chapter 5]). \square

Proof of Theorem 3.3. First we show that the required factorizations exist. Let $f: X \rightarrow Y$ be a map of spaces. By restricting to components if necessary, we may assume that X is 0-connected, and we fix a basepoint $x \in X$. Let P denote the maximal perfect subgroup of $\ker(\pi_1(f, x))$. Using the universal property of the plus construction with respect to P , as a perfect normal subgroup of $\pi_1(X, x)$, we have a factorization in \mathcal{S} as follows,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow i & \nearrow p \\ & & X_P^+ \end{array}$$

where $i \in \mathcal{A}$. Moreover, $p \in \mathcal{B}$ because there is an isomorphism

$$\ker(\pi_1(p, i(x))) \cong \ker(\pi_1(f, x))/P$$

and this group has no non-trivial perfect subgroups.

Next we show that $\mathcal{B} \subseteq \mathcal{A}^\perp$ (or $\mathcal{A} \subseteq {}^\perp\mathcal{B}$). Consider a square in \mathcal{S}

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

where $i \in \mathcal{A}$ and $p \in \mathcal{B}$. By restricting to components if necessary, we may assume that i is a map between 0-connected spaces. In this case, $B \simeq A_P^+$ for some perfect normal subgroup $P \trianglelefteq \pi_1(A, a)$, for some chosen basepoint $a \in A$. We obtain the following diagram in \mathcal{S}

$$\begin{array}{ccccc} \mathrm{map}_{\mathcal{S}}(B, X) & \xrightarrow{c} & E & \xrightarrow{j} & \mathrm{map}_{\mathcal{S}}(A, X) \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{map}_{\mathcal{S}}(B, Y) & \xrightarrow{i^*} & \mathrm{map}_{\mathcal{S}}(A, Y) \end{array}$$

where E denotes the pullback. By Lemma 3.4, the maps i^* and jc are monomorphisms. Hence j , and as a consequence c , are also monomorphisms. Therefore it suffices to show that c induces a bijection on π_0 . By Lemma 3.4, we know that the subset

$$\pi_0(E) \hookrightarrow \pi_0(\mathrm{map}_{\mathcal{S}}(A, X))$$

consists of those classes $g: A \rightarrow X$ such that P is contained in $\ker(\pi_1(pg, a))$. Since $p \in \mathcal{B}$, the image of the subgroup P in $\pi_1(X, g(a))$ must be trivial, hence P is actually contained in $\ker(\pi_1(g, a))$. This identifies it with the subset

$$\pi_0(\mathrm{map}_{\mathcal{S}}(B, X)) \hookrightarrow \pi_0(\mathrm{map}_{\mathcal{S}}(A, X)),$$

as was required to show.

Now we prove that $\mathcal{A}^\perp \subseteq \mathcal{B}$. Suppose that $p: X \rightarrow Y$ is in \mathcal{A}^\perp . Consider the factorization constructed above, $X \xrightarrow{i} \tilde{X} \xrightarrow{q} Y$, where $i \in \mathcal{A}$ and $q \in \mathcal{B}$. Then the square

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ i \downarrow & \nearrow & \downarrow p \\ \tilde{X} & \xrightarrow{q} & Y \end{array}$$

shows that i admits a retraction, and therefore i is an equivalence – alternatively, note that p is a retract of q . Lastly, we show that ${}^\perp\mathcal{B} \subseteq \mathcal{A}$. Suppose that $i: A \rightarrow B$ is a map in ${}^\perp\mathcal{B}$. We consider again the factorization constructed above, $A \xrightarrow{j} \tilde{B} \xrightarrow{p} B$, where $j \in \mathcal{A}$ and $p \in \mathcal{B}$, and the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{j} & \tilde{B} \\ i \downarrow & \nearrow l & \downarrow p \\ B & \xlongequal{\quad} & B \end{array}$$

The lift l shows that i is a retract of the map j , hence it is also an acyclic map – it also follows that l must be an equivalence. This completes the proof that $(\mathcal{A}, \mathcal{B})$ is a factorization system. The factorization system $(\mathcal{A}, \mathcal{B})$ defines a modality because acyclic maps are closed under pullbacks by definition. \square

Remark 3.5. The modality of Theorem 3.3 can be regarded as an instance of the construction in [2, Example 3.5.3] for the nullification that defines the plus construction (see [11, 3]). We emphasize that the modality of Theorem 3.3 is different from the factorization system that arises from the plus construction as a localization in the ∞ -category of spaces and whose left class is the class of maps which become equivalences after plus construction.

3.3. Blakers-Massey theorems for acyclic maps. As an application of the main results of [2], we obtain a Blakers-Massey theorem and a dual Blakers-Massey theorem associated with the modality $(\mathcal{A}, \mathcal{B})$ in \mathcal{S} .

Given a map $f: X \rightarrow Y$ and $y \in Y$, we write $F_{f,y}$ for the (homotopy) fiber of f at y . The Generalized Blakers-Massey theorem [2, Theorem 4.1.1] specialized to the modality $(\mathcal{A}, \mathcal{B})$ gives the following statement (see also the comments in [2, p. 30]): Given a pushout square in \mathcal{S}

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ f \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

such that for every $a \in A$, the map

$$(8) \quad F_{f,f(a)} \vee_a F_{g,g(a)} \rightarrow F_{f,f(a)} \times F_{g,g(a)}$$

is acyclic, then the canonical map $A \rightarrow B \times_D C$ is acyclic.

This statement can be reduced to a simpler statement as follows. First note that the fiber of the map (8) at the point (a', a'') is given by the join

$$\Omega_{a,a'} F_{f,f(a)} * \Omega_{a,a''} F_{g,g(a)}.$$

(Here $\Omega_{x,x'} X$, for $x, x' \in X$, denotes the pullback of $(* \xrightarrow{x} X \xleftarrow{x'} *)$.) We recall the following fact about the join construction.

Lemma 3.6. *Let X and Y be non-empty spaces in \mathcal{S} . Then $X * Y$ is contractible if and only if it is acyclic.*

Proof. Suppose that $X * Y$ is acyclic. Since $\tilde{H}_0(X; \mathbb{Z}) \otimes \tilde{H}_0(Y; \mathbb{Z}) \cong H_1(X * Y; \mathbb{Z}) = 0$, it follows that either X or Y is 0-connected. Then, by the van Kampen theorem applied iteratively, $X * Y$ is also 1-connected (see also [9, Lemma 2.2]). Hence $X * Y$ is contractible. \square

This leads to the following simplification of the assumption about the acyclicity of the map (8).

Proposition 3.7. *The map (8) is acyclic if and only if it is an equivalence.*

Proof. Suppose that (8) is acyclic. Using the description of the fibers of (8) in terms of the join construction, it follows that at most one of the fibers, $F_{f,f(a)}$ and $F_{g,g(a)}$, can fail to be 0-connected. If one of them is not 0-connected, say $F_{f,f(a)}$, then by choosing (a', a'') appropriately, it follows that the other fiber $F_{g,g(a)}$ must be contractible – and so the fibers of (8) are contractible as well. If both fibers are 0-connected, then the previous lemma applies. \square

Thus, the Blakers-Massey theorem for acyclic maps as stated above, reduces to the Little Blakers-Massey theorem [2, Corollary 4.1.4] specialized to \mathcal{S} : this asserts that if (8) is an equivalence for each $a \in A$, then the pushout square is also a pullback.

Remark 3.8. The join $X * Y$ of two (non-empty) spaces is acyclic if and only if there is a set of primes P such that one of the homologies, $\tilde{H}_*(X; \mathbb{Z})$ and $\tilde{H}_*(Y; \mathbb{Z})$, is P -torsion and the other is uniquely P -divisible. See [1, proof of Theorem 2.5].

Example 3.9. Let (X, x) and (Y, y) be based 0-connected spaces such that the join $\Omega_x X * \Omega_y Y$ is acyclic/contractible. Then an application of the Little Blakers-Massey theorem to the pushout square in \mathcal{S}

$$\begin{array}{ccc} X \times Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X * Y \end{array}$$

shows that the square is also a pullback. (Further comments: the Blakers-Massey theorem for acyclic maps yields that the canonical map $X \times Y \rightarrow X \times_{X * Y} Y$ is acyclic. But either $\Omega_x X$ or $\Omega_y Y$ must be 0-connected. As a consequence, the canonical map to this pullback is π_1 -injective and therefore it is an equivalence.)

Example 3.10. Let (X, x) and (Y, y) be based spaces such that $\pi_1(X, x)$ and $\pi_1(Y, y)$ are trivial, and the join $\Omega_x^2 X * \Omega_y^2 Y$ is acyclic/contractible. Then an application of the Little Blakers-Massey theorem to the pushout square

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee Y \end{array}$$

shows that it is also a pullback.

On the other hand, the dual Generalized Blakers-Massey theorem [2, Theorem 3.6.1] for the modality $(\mathcal{A}, \mathcal{B})$ specializes to the following statement: Given a pullback square in \mathcal{S}

$$(9) \quad \begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow f \\ B & \xrightarrow{g} & D \end{array}$$

such that that for every $d, d' \in D$, the join $F_{f,d} * F_{g,d'}$ is acyclic, then it follows that the canonical map $B \cup_A C \rightarrow D$ is acyclic.

For the proof, note that the fiber of this canonical map at $d \in D$ is exactly the join $F_{f,d} * F_{g,d}$ – this observation was also used in the proof of Theorem 2.1(b). Note that the assumption on the join is satisfied when f or g is acyclic.

In the case of 0-connected spaces, this statement is also part of [1, Theorem 2.5] in which case it is shown that the canonical map $B \cup_A C \rightarrow D$ is actually an equivalence. The same conclusion holds more generally when the maps f and g in (9) are in addition π_0 -surjective, so that their fibers are non-empty and Lemma 3.6 applies – in this case, the statement above becomes an instance of the dual Little

Blakers-Massey theorem. Moreover, this observation also gives a different proof of the fact that acyclic maps are balanced. On the other hand, the pullback square

$$\begin{array}{ccc} \emptyset & \longrightarrow & C \\ \downarrow & & \downarrow f \\ \emptyset & \longrightarrow & D \end{array}$$

where f is acyclic shows that the conclusion about acyclicity cannot be improved in general.

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