SEMINAR NOTES ON ∞-TOPOI

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1. Preliminaries

A functor $F : \mathscr{C} \to \mathscr{D}$ between ∞-categories is a **localization** if it has a fully faithful right adjoint G. Such an adjunction is determined by the endofunctor $L := GF : \mathscr{C} \to \mathscr{C}$ and the natural transformation $\alpha : 1_{\mathscr{C}} \to L$ which has the property that $L(\alpha_C), \alpha_{L(C)} : L(C) \to L^2(C)$ are equivalences. Given such a pair (L, α), $\mathscr{D}$ is equivalent to the essential image of L. See [HTT, 5.2.7].

A localization $F : \mathscr{C} \to \mathscr{D}$ is also determined by the class of morphisms $S_F := \{f \in \mathscr{C} \mid F(f) \text{ is an equivalence}\}$. The ∞-category $\mathscr{D}$ is equivalent to the full subcategory of $\mathscr{C}$ spanned by the **S_F-local objects**, that is, objects $X \in \mathscr{C}$ such that

$$\mathrm{map}_{\mathscr{C}}(Z, X) \xrightarrow{\simeq} \mathrm{map}_{\mathscr{C}}(Y, X)$$

for any morphism $f : Y \to Z$ in S_F. Assuming that $\mathscr{C}$ has small colimits, the class S_F is **strongly saturated**: it is closed under pushouts in $\mathscr{C}$, it is closed under colimits in $\mathscr{C}^{\to}$, and has the 2-out-of-3 property. See [HTT, 5.5.4].

Let $\mathscr{C}$ be a presentable ∞-category and S a set of morphisms. We denote $\overline{S}$ the smallest strongly saturated class of morphisms which contains S. Let $\mathscr{D}$ be the full subcategory of S-local objects. Then the inclusion $\mathscr{D} \subseteq \mathscr{C}$ has a left adjoint $F : \mathscr{C} \to \mathscr{D}$ which is a localization. As a consequence, $\mathscr{D}$ is presentable. The class of morphisms which become equivalences is exactly $\overline{S}$, i.e., $S_F = \overline{S}$. Lastly, every accessible localization of $\mathscr{C}$ arises from a set of morphisms S in this way. See [HTT, 5.5.4].

2. Left exact localizations

A localization $F : \mathscr{C} \to \mathscr{D}$ is **left exact** if F preserves finite limits. Assuming that $\mathscr{C}$ has finite limits, a localization F is left exact if and only if the class S_F is closed under pullbacks in $\mathscr{C}$ [HTT, 6.2.1.1].

We can try to construct left exact accessible localizations as follows. Let $\mathscr{C}$ be a presentable ∞-category and S a set of morphisms. Let $\widetilde{S}$ denote the smallest strongly saturated class in $\mathscr{C}$ which is closed under pullbacks in $\mathscr{C}$ and contains S. Then there is an accessible localization $F : \mathscr{C} \to \mathscr{D}$ such that $S_F = \widetilde{S}$ if and only if $\widetilde{S}$ is generated by a set of morphisms as a strongly saturated class. Such a set exists if colimits in $\mathscr{C}$ are universal and pullbacks commute with filtered colimits [HTT, 6.2.1.2].

An ∞-category $\mathscr{X}$ is an **∞-topos** if there exists a small ∞-category $\mathcal{C}$ and an accessible left exact localization $\mathcal{P}(\mathcal{C}) \to \mathscr{X}$. ($\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ is the ∞-category of presheaves on $\mathcal{C}$. $\mathcal{S}$ denotes the ∞-category of spaces.)

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3. Descent Properties

Let $\mathscr{C}$ be a presentable ∞-category and K a simplicial set. Consider two diagrams

$$\overline{p}, \overline{q} : K^{\triangleright} \to \mathscr{C}$$

where $\overline{q}$ is a colimit diagram, and $\overline{\alpha} : \overline{p} \to \overline{q}$ a natural transformation whose restriction $\alpha = \overline{\alpha}_{|K}$ to K is **cartesian**, i.e. for each $x \to y$ in K, the square

$$\begin{array}{ccc} \overline{p}(x) & \longrightarrow & \overline{p}(y) \\ \downarrow & & \downarrow \\ \overline{q}(x) & \longrightarrow & \overline{q}(y) \end{array}$$

is a pullback. The ∞-category $\mathscr{C}$ satisfies **descent** if for all such quadruples $(K, \overline{p}, \overline{q}, \overline{\alpha})$, the following hold:

(D1) If $\overline{\alpha}$ is cartesian, then $\overline{p}$ is a colimit diagram.
(D2) If $\overline{p}$ is a colimit diagram, then $\overline{\alpha}$ is cartesian.

The first property (D1) says that colimits in $\mathscr{C}$ are universal. The second property (D2) says that the extension of a cartesian transformation to the colimit diagrams is again cartesian - a kind of "gluing" property for cartesian transformations.

It is a consequence of classical results that the ∞-category $\mathcal{S}$ of spaces satisfies descent. From this follows that $\mathcal{P}(\mathcal{C})$ satisfies descent for any small ∞-category $\mathcal{C}$. The property of descent is preserved under left exact localizations, therefore every ∞-topos satisfies descent too. See [HTT, 6.1.3]. An analogous study of descent in the context of model categories can be found in [T&HT].

4. Giraud Axioms

Let $\mathscr{C}$ be a presentable ∞-category which satisfies descent. In particular, it has the following properties:

- Colimits in $\mathscr{C}$ are universal. This means that for each $f : X \to Y$, the pullback functor $f^* : \mathscr{C}_{/Y} \to \mathscr{C}_{/X}$ preserves colimits. This is a consequence descent property (D1). See [HTT, 6.1.3.3].
- Coproducts $\mathscr{C}$ are disjoint. This means that the squares

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \sqcup Y \end{array}$$

  are pullbacks. This is a consequence of the descent properties (D1) and (D2). See [HTT, 6.1.3.19].
- Groupoids in $\mathscr{C}$ are effective. A **groupoid** in $\mathscr{C}$ is a simplicial object $U_\bullet : N(\Delta^{op}) \to \mathscr{C}$ such that

$$\begin{array}{ccc} U_n & \longrightarrow & U_S \\ \downarrow & & \downarrow \\ U_{S'} & \longrightarrow & U_0 \end{array}$$

  is a pullback for all decompositions $[n] = S \cup S'$ with $S \cap S' = \{s\}$. A groupoid object $U_\bullet$ is called **effective** if it can be extended to a colimit diagram, given by an augmented simplicial object $U_\bullet^+ : N(\Delta_+^{op}) \to \mathscr{C}$, such that

$$\begin{array}{ccc} U_1 & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ U_0 & \longrightarrow & U_{-1} \end{array}$$

is a pullback. Note that then the simplicial object is determined from the morphism $f : U_0 \to U_{-1}$ by taking iterated pullbacks. This type of augmented simplicial object is called the **Čech nerve of** f. See [HTT, 6.1.2.7-6.1.2.15].

The fact that groupoids in $\mathscr{C}$ are effective is a consequence of descent property (D2). See [HTT, 6.1.3.19].

Conversely, these properties characterize ∞-topoi.

Theorem 1. *Let $\mathscr{X}$ be an ∞-category. The following are equivalent:*

(1) *$\mathscr{X}$ is an ∞-topos.*
(2) *$\mathscr{X}$ is a presentable ∞-category and satisfies descent.*
(3) *$\mathscr{X}$ satisfies the following:*
  (a) *$\mathscr{X}$ is a presentable ∞-category.*
  (b) *Colimits in $\mathscr{X}$ are universal.*
  (c) *Coproducts in $\mathscr{X}$ are disjoint.*
  (d) *Groupoids in $\mathscr{X}$ are effective.*

Proof. (3) $\Rightarrow$ (1): Let $\mathscr{X}_\kappa$ be the small full subcategory of $\mathscr{X}$ spanned by the κ-compact objects. We may choose κ so that $\mathscr{X}$ is κ-presentable and κ-compact objects are closed under pullbacks. Then we have an accessible localization $F : \mathcal{P}(\mathscr{X}_\kappa) \to \mathscr{C}$ and it suffices to show that it is left exact. By assumption, the restriction of F along the Yoneda embedding $j : \mathscr{X}_\kappa \hookrightarrow \mathcal{P}(\mathscr{X}_\kappa)$ is left exact by construction. It is a consequence of (3)(b)-(d), that the localization functor is again left exact. Using (3)(b), we can reduce this claim to showing that F preserves pullbacks of the form

$$\begin{array}{ccc} W & \longrightarrow & j(C) \\ \downarrow & & \downarrow \\ j(C') & \longrightarrow & Z \end{array}$$

Since this holds when Z is representable, it suffices to show that the class of objects Z for which this holds is closed under colimits. (3)(c) is used to show that it is closed under coproducts and (3)(d) is used to show that it is closed under coequalizers. See [HTT, 6.1.5.2-6.1.5.3]. An analogous Giraud-type characterization in the context of model categories is shown in [HAGI, 4.9.2]. □

5. Monomorphisms and Effective Epimorphisms

Let $\mathscr{C}$ be an ∞-category. A morphism $f : X \to Z$ is a **monomorphism** if for each object $Y \in \mathscr{C}_{/Z}$, the mapping space $\mathrm{map}_{\mathscr{C}_{/Z}}(X, Y)$ is either empty or contractible. Equivalently, for any $Y \in \mathscr{C}$, the map $\mathrm{map}_{\mathscr{C}}(Y, X) \to \mathrm{map}_{\mathscr{C}}(Y, Z)$ is up to weak equivalence an inclusion of path components. Monomorphisms are closed under pullbacks. A morphism $f : X \to Z$ is a monomorphism if and only if $X \xrightarrow{\simeq} X \times_Z X$.

Let $\mathscr{C}$ be a presentable ∞-category and $X \in \mathscr{C}$. Let $\mathrm{Sub}(X)$ denote the class of equivalence classes of monomorphisms $U \to X$. This is a (small) poset which is locally presentable as an (ordinary) category. See [HTT, 6.2.1.3].

Let $\mathscr{X}$ be an ∞-topos. A morphism $f : U \to X$ is an **effective epimorphism** if the Čech nerve $\check{C}(f)$, i.e. the augmented simplicial object which is defined by iterated pullbacks along the morphism f, is a **simplicial resolution** of X, i.e. a colimit diagram.

A morphism $f : U \to X$ is an effective epimorphism if and only if $f^* : \mathrm{Sub}(X) \to \mathrm{Sub}(U)$ is injective. The class of effective epimorphisms contains the equivalences, it is closed under composition, and it is closed under coproducts. Furthermore, if gf is an effective epimorphism, then so is g. A morphism is an equivalence if and only if it is a monomorphism and an effective epimorphism. Left exact colimit-preserving functors preserve effective monorphisms. In the ∞-category of spaces, a map is an effective epimorphism if and only if it is 0-connected. See [HTT, 6.2.3].

Proposition 2. *Let $\mathscr{X}$ be an ∞-topos, $f : V \to X$ a morphism, $V_\bullet$ the associated Čech nerve, and $|V_\bullet|$ the colimit of the underlying simplicial object. Then*

$$\begin{array}{ccc} V & \xrightarrow{\;p\;} & |V_\bullet| \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle j} & \\ & X & \end{array}$$

is a factorization of f into an effective epimorphism p and a monomorphism j. Moreover, j is the (-1)-truncation of f in $\mathscr{X}_{/X}$.

Proof. The morphism p is an effective epimorphism because $\mathscr{X}$ is an ∞-topos and therefore the underlying groupoid of $V_\bullet$ is effective. Consider the pullback squares

$$\begin{array}{ccc} V_n \times_{|V_\bullet|} V_m & \longrightarrow & V_n \times_X V_m \\ \downarrow & & \downarrow \\ V \times_{|V_\bullet|} V & \longrightarrow & V \times_X V \end{array}$$

The bottom morphism is an equivalence since both objects are equivalent to V_1. Hence the top morphism is an equivalence for all $m, n \geq 0$. As a consequence, $|V_\bullet| \simeq |V_\bullet| \times_{|V_\bullet|} |V_\bullet| \xrightarrow{\simeq} |V_\bullet| \times_X |V_\bullet|$ and therefore j is a monomorphism. For every other monomorphism $j' : \widetilde{V} \to X$ that factors f, there is a homotopically unique factorization through $|V_\bullet|$ provided by $|V_\bullet| \to |\check{C}(j')| \simeq \widetilde{V}$. See [HTT, 6.2.3.4]. $\square$

6. Topological Localizations

Let $\mathscr{C}$ be a presentable ∞-category. A strongly saturated class S is called **topological** if: (a) it is generated as a strongly saturated class by monomorphisms, and (b) it is closed under pullbacks. Note that the definition does *not* require that the class is generated by a *set* of monomorphisms. But this is always true if colimits in $\mathscr{C}$ are universal as then any monomorphism in S is the colimit of monomorphisms in S whose codomains belong to a set of objects that generate $\mathscr{C}$ under colimits. See [HTT, 6.2.1.5].

A localization $F : \mathscr{C} \to \mathscr{D}$ is called **topological** if the class of morphisms S_F is topological.

Proposition 3. *Every topological localization $F : \mathcal{P}(\mathcal{C}) \to \mathscr{D}$ is accessible and left exact.*

7. Grothendieck Topologies

Let $\mathcal{C}$ be an ∞-category. A **sieve on** $\mathcal{C}$ is a full subcategory $U \subseteq \mathcal{C}$ such that if $f : C \to C'$ is in $\mathcal{C}$ and $C' \in U$, then f is in U. There is bijection between the collection of sieves on $\mathcal{C}$ and the equivalence classes of (-1)-truncated object in $\mathcal{P}(\mathcal{C})$. Given a (-1)-truncated presheaf F, the associated sieve is spanned by the objects $C \in \mathcal{C}$ such that $F(C) \neq \varnothing$.

A **sieve on** $C \in \mathcal{C}$ is a sieve on the ∞-category $\mathcal{C}_{/C}$. Accordingly, there is a bijection between the collection of sieves on C and $\mathrm{Sub}(j(C))$, the equivalence classes of subobjects of the representable functor of C in $\mathcal{P}(\mathcal{C})$ - this is the same as equivalence classes of (-1)-truncated objects in $\mathcal{P}(\mathcal{C}_{/C})$. See [HTT, 6.2.2.5].

A **Grothendieck topology on** $\mathcal{C}$ consists of a collection of sieves on $C \in \mathcal{C}$ for each C, called **covering sieves**, such that:

(1) $\mathcal{C}/C \subseteq \mathcal{C}/C$ is a covering sieve for each C.
(2) Given $f : D \to C$ and a covering sieve $U \subseteq \mathcal{C}/C$, then $f^*U \subseteq \mathcal{C}/D$ is a covering sieve. (f^*U is spanned by the objects $T \to D$ such that $(U \to D \xrightarrow{f} C) \in U$.)
(3) Given $C \in \mathcal{C}$ and a covering sieve $U \subseteq \mathcal{C}/C$, a sieve $U' \subseteq \mathcal{C}/C$ is covering if f^*U' is a covering sieve for each $f : D \to C$ in U.

There is a bijection between Grothendieck topologies on $\mathcal{C}$ and Grothendieck topologies on the (ordinary) homotopy category $h\mathcal{C}$. This is because the canonical functor $h(\mathcal{C}/C) \to h(\mathcal{C})/C$ is full and therefore it induces a bijection between sieves on $C \in \mathcal{C}$ and sieves on $C \in h(\mathcal{C})$.

Let $(\mathcal{C}, \tau)$ be a small ∞-category equipped with a Grothendieck topology, denoted by τ. Let S_τ denote the collection of monorphisms $U \to j(C)$ in $\mathcal{P}(\mathcal{C})$ which correspond to covering sieves. A presheaf $F \in \mathcal{P}(\mathcal{C})$ is called a **sheaf** if it is S_τ-local. The full subcategory of sheaves is denoted by $\mathrm{Sh}(\mathcal{C}, \tau)$.

Theorem 4. *Let $\mathcal{C}$ be a small ∞-category.*

(a) *Let τ be a Grothendieck topology on $\mathcal{C}$. Then $\mathrm{Sh}(\mathcal{C}, \tau)$ is a topological localization of $\mathcal{P}(\mathcal{C})$.*
(b) *There is a bijection between Grothendieck topologies on $\mathcal{C}$ and equivalence classes of topological localizations of $\mathcal{P}(\mathcal{C})$.*

Proof. (a) The localization functor is given by sheafification: for each presheaf F, the sheafification LF is given by a transfinite application of an operation $F \mapsto F^+$ which replaces $F(C)$, $C \in \mathcal{C}$, with the colimit over all covering sieves $U \subseteq \mathcal{C}/C$ of the limit of F restricted to U. The number of times this operation must be applied depends on the homotopical sizes of the covering sieves. This is analogous to the definition of sheafification in ordinary topos theory. There is a canonical natural transformation $F \to F^+$ which is an S_τ-local equivalence. The sheafification functor is left exact because the collection of covering sieves is closed under intersections. See [HTT, 6.2.2.7].

(b) The inverse is defined as follows: given a topological localization $F : \mathcal{P}(\mathcal{C}) \to \mathscr{D}$, a monomorphism $U \to j(C)$ is defined to be (or corresponds to) a covering sieve if it is in S_F. This defines a Grothendieck topology because the localization is left exact. When $\mathscr{D} = \mathrm{Sh}(\mathcal{C}, \tau)$, this Grothendieck topology is exactly τ. See [HTT, 6.2.2.17]. □

The next result identifies the universal property of the ∞-category of sheaves $\mathrm{Sh}(\mathcal{C}, \tau)$.

Proposition 5. *Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology τ and $L : \mathcal{P}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}, \tau)$ the associated accessible left exact localization. Let $\mathscr{X}$ be an ∞-topos. Then the composition*

$$\mathrm{Fun}^*(\mathrm{Sh}(\mathcal{C}, \tau), \mathscr{X}) \xrightarrow{L^*} \mathrm{Fun}^*(\mathcal{P}(\mathcal{C}), \mathscr{X}) \xrightarrow{j^*} \mathrm{Fun}(\mathcal{C}, \mathscr{X})$$

is fully faithful. Here Fun^* denotes the ∞-category of left exact colimit-preserving functors.

If $\mathcal{C}$ has finite limits, then $f : \mathcal{C} \to \mathscr{X}$ is in the essential image if and only if

(a) *f is left exact, and*
(b) *for every collection $\{C_\alpha \to C\}_\alpha$ which generates a covering sieve, the morphism*

$$\coprod_\alpha f(C_\alpha) \to f(C)$$

is an effective epimorphism.

Proof. The functor L^* is well-defined because L is left exact. It is fully faithful because L is a localization. j^* is fully faithful because it is the restriction of the equivalence between all colimit-preserving functors $\mathcal{P}(\mathcal{C}) \to \mathscr{X}$ and functors $\mathcal{C} \to \mathscr{X}$ - by the universal property of the ∞-category of presheaves. See [HTT, 5.1.5 and 5.2.7.12].

For the second part, suppose that $f : \mathcal{C} \to \mathscr{X}$ is the restriction of $\mathcal{P}(\mathcal{C}) \xrightarrow{L} \mathrm{Sh}(\mathcal{C}, \tau) \xrightarrow{F} \mathscr{X}$. Since the Yoneda embedding is left exact, f is also left exact so (a) is satisfied. For (b), it suffices to show that

$$\coprod_\alpha L(j(C_\alpha) \to L(j(C))$$

is an effective epimorphism - since F preserves effective epimorphisms. Consider the factorization

$$\coprod_\alpha j(C_\alpha) \xrightarrow{p} U \xrightarrow{i} j(C)$$

into an effective epimorphism p and a monomorphism i. Then i can be identified with the covering sieve that the collection $\{C_\alpha \to C\}$ generates. Then $L(p)$ is an effective epimorphism and $L(i)$ is an equivalence.

The converse is similar.

See [HTT, 6.2.3.20]. □

8. Is every ∞-topos an ∞-category of sheaves?

Let $\mathscr{X}$ be an ∞-topos. We may choose κ so that $\mathscr{X}$ is κ-presentable and the full subcategory $\mathscr{X}_\kappa$ of κ-compact objects is closed under pullbacks. Then there is a left exact accessible localization $F : \mathcal{P}(\mathscr{X}_\kappa) \to \mathscr{X}$. We can consider the largest Grothendieck topology on $\mathscr{X}_\kappa$ which is compatible with this localization: a sieve on $C \in \mathscr{X}_\kappa$, $U \to j(C)$, is covering if $F(U) \to F(j(C))$ is an equivalence in $\mathscr{X}$.

This defines a Grothendieck topology τ_F on $\mathscr{X}_\kappa$ - it is an example of a **canonical topology** [HTT, 6.2.4]. Moreover, F descends to the ∞-category of sheaves,

$$\begin{array}{ccc} \mathcal{P}(\mathscr{X}_\kappa) & & \\ {\scriptstyle L}\downarrow & \searrow^{F} & \\ \mathrm{Sh}(\mathcal{C}, \tau_F) & \overset{\bar{F}}{\dashrightarrow} & \mathscr{X} \end{array}$$

but unlike the case of ordinary Grothendieck topoi (or n-topoi for $n < \infty$), the induced functor $\bar{F}$ is not an equivalence in general.

Since $\bar{F}$ is left exact, it induces a functor on the full (reflective) subcategories of k-truncated objects, for $-1 \leq k < \infty$,

$$\mathcal{P}(\mathscr{X}_\kappa)_{\leq k} \rightleftarrows \mathrm{Sh}(\mathscr{X}_\kappa, \tau_F)_{\leq k} \overset{\bar{F}_{\leq k}}{\rightleftarrows} \mathscr{X}_{\leq k}.$$

Recall that an object X in an ∞-category $\mathcal{C}$ is k**-truncated** if for each A in $\mathcal{C}$, the space $\mathrm{map}_{\mathcal{C}}(A, X)$ has trivial homotopy groups in degrees larger than k for each basepoint. A presheaf $F \in \mathcal{P}(\mathcal{C})$ is k-truncated if and only if its values are k-truncated spaces. See [HTT, 5.5.6].

The induced functor $\bar{F}_{\leq k}$ is an equivalence, i.e. the ∞-categories of k-truncated objects $\mathscr{X}_{\leq k}$, are equivalent to ∞-categories of k-truncated objects in ∞-categories of sheaves. To see this, we show inductively that if $f : X \to Y$ is l-truncated, i.e. the homotopy fibers of $\mathrm{map}_{\mathcal{C}}(A, X) \to \mathrm{map}_{\mathcal{C}}(A, Y)$ are l-truncated for all A, and $\bar{F}(f)$ is an equivalence, then f is an equivalence. For $l = -1$, i.e. when f is a monomorphism, this is true by construction. For the inductive step, note that f is l-truncated if and only if $X \to X \times_Y X$ is $(l-1)$-truncated. If $\bar{F}(f)$ is an equivalence, then $\bar{F}(X \to X \times_Y X)$ is an equivalence, since $\bar{F}$ is left exact. Hence, by the inductive assumption, $X \to X \times_Y X$ is an equivalence, which implies that f is an equivalence. See [HTT, 6.4.1.6].

The ∞-category $\mathscr{X}_{\leq k}$ is a $(k+1)$**-topos** - this can be used as a definition. There are several different characterizations of n-topoi, for $0 \leq n < \infty$, including a Giraud-type characterization similar to Theorem 1. See [HTT, 6.4.1.5].

9. Homotopy groups and Hypercompetions

Let $\mathscr{X}$ be an ∞-topos and $X \in \mathscr{X}$. There is an associated object X^{S^n} together with a morphism $s : X^{S^n} \to X$ given by "evaluation" at the basepoint of S^n. X^{S^n} is specified by the property that $\mathrm{map}_{\mathscr{X}}(Y, X^{S^n}) \simeq \mathrm{map}(S^n, \mathrm{map}_{\mathscr{X}}(Y, X))$ for all $Y \in \mathscr{X}$. The n**-th homotopy group** $\pi_n(X)$ of X is the 0-truncation $\tau_{\leq 0}(s)$ in the ∞-category $\mathscr{X}/X$. This is a group object if $n > 0$ and commutative if $n > 1$. More generally, it is possible to define relative homotopy groups and obtain long exact sequences as in the classical context of topological spaces. In fact, if $\mathscr{X} = \mathcal{S}$ and $x : * \to X$ is a basepoint, then $x^*(\pi_n(X))$ is $\pi_n(X, x)$. Homotopy groups are preserved along geometric morphisms, i.e. left exact left adjoints, because these preserve truncated objects. See [HTT, 6.5.1].

If an object $X \in \mathscr{X}$ is n-truncated, then $\pi_k(X) \simeq *$ for all $k > n$. Moreover, if $n \geq 0$ and $\pi_n(X) \simeq *$, then X is $(n-1)$-truncated. See [HTT, 6.5.1.7].

A morphism $f : X \to Y$ in $\mathscr{X}$ is n**-connective**, $0 \leq n \leq \infty$, if it is an effective epimorphism and $\pi_k(f) \simeq *$ for $0 \leq k \leq n$. A morphism $f : X \to Y$ is n-connective if and only if $X \to X \times_Y X$ is $(n-1)$-connective and f is an effective epimorphism. See [HTT, 6.5.1.16-6.5.1.18]. An n-truncated ∞-connective morphism is an equivalence.

The class $\mathcal{W}_\infty$ of ∞-connective morphisms is a strongly saturated class generated by a set of morphisms. This is because it can be specified by accessible conditions. See [HTT, 6.5.2.8]. Furthermore, it is closed under pullbacks. An object $X \in \mathscr{X}$ is **hypercomplete** if it is $\mathcal{W}_\infty$-local. This is equivalent to saying that X satisfies descent with respect to the **hypercoverings** [HTT, 6.5.3.13], i.e. X is local with respect to the morphisms $\{|U_\bullet| \to X\}$ for each hypercovering $U_\bullet$ in $\mathscr{X}/X$. See [HTT, 6.5.3] - a discussion of the comparison between descent with respect to coverings and hypercoverings can be found in [HTT, 6.5.4].

The full subcategory of hypercomplete objects $\mathscr{X}^\wedge \subseteq \mathscr{X}$ admits a left exact accessible localization, called **hypercompletion**, $\mathscr{X} \to \mathscr{X}^\wedge$ and therefore $\mathscr{X}^\wedge$ is again an ∞-topos. $\mathscr{X}$ is **hypercomplete** if $\mathscr{X} = \mathscr{X}^\wedge$. $\mathscr{X}^\wedge$ is hypercomplete. The local model structure on simplicial presheaves due to Jardine [Jar], where the weak equivalences between simplicial presheaves are detected by the sheaves of homotopy groups, is hypercomplete. See [HTT, 6.5.2.14].

Proposition 6. *Let $\mathscr{X}$ be an ∞-topos, $(\mathcal{C}, \tau)$ a small ∞-category equipped with a Grothendieck topology τ, and $F : \mathrm{Sh}(\mathcal{C}, \tau) \to \mathscr{X}$ a left exact localization. Suppose that for every monomorphism u in $\mathrm{Sh}(\mathcal{C}, \tau)$, if $F(u)$ is an equivalence in $\mathscr{X}$, then u is an equivalence in $\mathrm{Sh}(\mathcal{C}, \tau)$. Then for every morphism $u \in \mathrm{Sh}(\mathcal{C}, \tau)$, if $F(u)$ is an equivalence in $\mathscr{X}$, then u is ∞-connective in $\mathrm{Sh}(\mathcal{C}, \tau)$.*

Proof. This is a special case of [HTT, 6.5.2.16]. Let $u : X \to Z$ be a morphism such that $F(u)$ is an equivalence. Using Proposition 2, we can find a factorization

$$X \xrightarrow{p} U \xrightarrow{i} Z$$

where p is an effective epimorphism and i is a monomorphism. It follows that $F(i)$ is an effective epimorphism and therefore an equivalence. Hence i is an equivalence, which implies that u is an effective epimorphism.

Proceeding by induction, suppose that for every morphism $u : X \to Z$ in $\mathrm{Sh}(\mathcal{C}, \tau)$, if $F(u)$ is an equivalence, then u is $(n-1)$-connective. For such a morphism u, the morphism $F(X \to X \times_Z X)$ is an equivalence, since F is left exact. Then $X \to X \times_Z X$ is $(n-1)$-connective and therefore u is n-connective, too. □

Combined with the discussion in Section 8, Theorem 6 implies that every ∞-topos is obtained from an ∞-category of sheaves by a left exact accessible localization at a collection of ∞-connective morphisms - this is a **cotopological** localization [HTT, 6.5.2.17]. The terminal such localization is given by the hypercompletion $\mathrm{Sh}(\mathcal{C}, \tau)^\wedge$.

Theorem 7. *Let $\mathcal{C}$ be a small ∞-category. There is a bijective correspondence between Grothendieck topologies on $\mathcal{C}$ and hypercomplete left exact accessible localizations of $\mathcal{P}(\mathcal{C})$.*

Proof. This is proved in [HAGI, 3.8.3] in the context of model categories. The bijective correspondence is analogous to the one in Theorem 4. □

As a consequence of Theorems 4 and 7, if $\mathrm{Sh}(\mathcal{C}, \tau)$ is not hypercomplete, then $\mathrm{Sh}(\mathcal{C}, \tau)^\wedge$ is not a topological localization of $\mathcal{P}(\mathcal{C})$.

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