

ON SERRE MICROFIBRATIONS AND A LEMMA OF M. WEISS

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ABSTRACT. We show that every fiberwise map from a Serre microfibration to a Serre fibration is n -connected if it is fiberwise n -connected. This generalizes a result of M. Weiss and related results by Bökstedt-Madsen and Galatius-Randal-Williams. We also discuss an application to configuration spaces.

1. INTRODUCTION

How can one determine the homotopy types of the homotopy fibers of a map $f : X \rightarrow Y$ from the homotopy types of its strict fibers? These are the same in the case where f is a (quasi-)fibration, but the comparison between the two types of fibers is a difficult task in general. Even in the case where all strict fibers are homotopy equivalent, more information is required in order to identify this common homotopy type with that of the homotopy fiber. The required information should provide a way of understanding how the individual fibers are organized collectively in a parametrized family of spaces. This information may be expressed by different types of properties: in terms of lifting properties, as in the case of Serre/Hurewicz fibrations, in terms of properties of f locally in Y , as in the case of local quasi-fibrations and related notions, or in terms of point-set topological assumptions on X , Y and f as, for example, in the case of results about cell-like maps.

In this note, we will be interested in the following notion from [10] which states a weak lifting property.

Definition 1.1. A map $p : E \rightarrow B$ is called a Serre microfibration if for any $k \geq 0$ and any commutative diagram

$$\begin{array}{ccc} \{0\} \times D^k & \xrightarrow{u} & E \\ \downarrow & & \downarrow p \\ [0, 1] \times D^k & \xrightarrow{v} & B \end{array}$$

there exists an $\epsilon > 0$ and a map $h : [0, \epsilon] \times D^k \rightarrow E$ such that $h(0, x) = u(x)$ and $p \circ h(t, x) = v(t, x)$ for all $x \in D^k$ and $t \in [0, \epsilon]$.

Example 1.2. Let $q : V \rightarrow B$ be a Serre fibration and $E \subset V$ an open subset. Then $q|_E : E \rightarrow B$ is a Serre microfibration. More generally, an open subspace of the total space of a Serre microfibration defines again a Serre microfibration.

Weiss' *microfibration lemma* [10, Lemma 2.2] states that a Serre microfibration $p : E \rightarrow B$ with weakly contractible fibers is actually a Serre fibration. As a consequence, it is also a weak homotopy equivalence. This result was generalized in

[3] as follows: if the fibers of p are n -connected, then p has the right lifting property with respect to the inclusions

$$(*_{\leq n}) \quad \{0\} \times D^k \rightarrow [0, 1] \times D^k, \quad k \leq n.$$

As a consequence, each map from a fiber to the homotopy fiber is n -connected, hence the homotopy fibers are n -connected, and the map p is $(n+1)$ -connected [3, Proposition 2.6].

The question of extending these results to arbitrary Serre microfibrations, whose fibers are abstractly weakly (or n -) homotopy equivalent, faces the problem of finding a uniform way of comparing them. In this direction, Bökstedt and Madsen [1, Lemma 3.6] proved, under certain point-set topological assumptions, that given a commutative diagram as follows, where the top map is an open inclusion and p is the projection,

$$\begin{array}{ccc} U & \xrightarrow{i} & X \times B \\ & \searrow & \swarrow p \\ & & B \end{array}$$

then i is a weak homotopy equivalence if for every $b \in B$, the map on fibers, $U \cap p^{-1}(b) \rightarrow X$, is a weak homotopy equivalence.

The statement below generalizes these results to the case of fiberwise n -connected maps with target an arbitrary fibration. We say that a map is a *Serre n -fibration* if it has the right lifting property with respect to the inclusions in $(*_{\leq n})$ above.

Theorem 1.3. *Let $p : E \rightarrow B$ be a Serre microfibration, $q : V \rightarrow B$ be a Serre fibration, and $f : E \rightarrow V$ a map over B . Suppose that $f_b : p^{-1}(b) \rightarrow q^{-1}(b)$ is $(n+1)$ -connected, $n \geq 0$, for all $b \in B$. Then p is a Serre n -fibration and, as a consequence, the map $f : E \rightarrow V$ is $(n+1)$ -connected.*

Remark 1.4. The statement is true, of course, for arbitrary maps $q : V \rightarrow B$ if we replace the fibers of q with the homotopy fibers.

The proof of Theorem 1.3 will be given in Section 2. In Section 3, we give an application to configuration spaces and discuss the connection with the context of abstract transversality that was used in [1] and [3].

2. THE PROOF OF THEOREM 1.3

The proof of Theorem 1.3 follows [10, Lemma 2.2] and [3, Proposition 2.6].

Let $I = [0, 1]$ be the unit interval. Let $X^I = \text{Map}(I, X)$ be the space of paths with the compact-open topology. The following observation is due to Weiss [10, Lemma 2.2].

Lemma 2.1. *Let $p : E \rightarrow B$ be a Serre microfibration. Then $p^I : E^I \rightarrow B^I$ is also a Serre microfibration.*

Proof. This follows directly by adjunction. □

We also recall the following lemma from [3, p. 8].

Lemma 2.2 (Galatius-Randal-Williams). *Let (Y, X) be a finite CW pair and $p : E \rightarrow B$ a Serre microfibration. Suppose we are given a lifting problem*

$$\begin{array}{ccc} X & \xrightarrow{u} & E \\ \downarrow & & \downarrow p \\ Y & \xrightarrow{v} & B \end{array}$$

If there exists a map $h : Y \rightarrow E$ lifting v and such that $h|_X$ is fiberwise homotopic to u , then there is also a map $h' : Y \rightarrow E$ that makes the diagram commute strictly.

Lemma 2.3. *Let p, q and f be as in Theorem 1.3. Then the map*

$$(f^I)_b : (p^I)^{-1}(b) \rightarrow (q^I)^{-1}(b)$$

is n -connected for all $b : I \rightarrow B$.

Proof. By adjunction, it suffices to prove that for each $k \leq n$ and each lifting problem

$$\begin{array}{ccccc} [0, 1] \times \partial D^k & \xrightarrow{u} & E & \xrightarrow{f} & V \\ \downarrow & \nearrow \bar{h} & \searrow h & & \downarrow q \\ [0, 1] \times D^k & \longrightarrow & [0, 1] & \xrightarrow{b} & B \end{array}$$

there exists a map $\bar{h} : [0, 1] \times D^k \rightarrow E$ over B so that the upper triangle commutes, and the triangle in the middle commutes up to fiberwise homotopy over B and relative to $[0, 1] \times \partial D^k$. For $t \in [0, 1]$, consider the restricted diagram

$$\begin{array}{ccccc} \{t\} \times \partial D^k & \longrightarrow & p^{-1}(b(t)) & \longrightarrow & E \\ \downarrow & \nearrow h'_t & \downarrow & & \downarrow f \\ \{t\} \times D^k & \longrightarrow & q^{-1}(b(t)) & \longrightarrow & V \xrightarrow{q} B \end{array}$$

where the lower composite is the constant map at $b(t) \in B$. By assumption, there exists a map $h'_t : \{t\} \times D^k \rightarrow p^{-1}(b(t))$ such that the upper triangle commutes and the lower triangle commutes up to homotopy relative to ∂D^k . The microfibration property of p applied to the commutative diagram

$$\begin{array}{ccc} \{t\} \times D^k \cup [0, 1] \times \partial D^k & \xrightarrow{h'_t \cup u} & E \\ \downarrow & & \downarrow p \\ [0, 1] \times D^k & \longrightarrow & B \end{array}$$

provides an extension of the lift $h'_t : \{t\} \times D^k \rightarrow p^{-1}(b(t)) \subseteq E$ to a map $h'_{t,\epsilon}$ defined on $[t - \epsilon(t), t + \epsilon(t)] \times D^k$, for some $\epsilon(t) > 0$, and such that the diagram commutes

$$\begin{array}{ccc} [t - \epsilon(t), t + \epsilon(t)] \times \partial D^k & \xrightarrow{u} & E \\ \downarrow & \nearrow h'_{t,\epsilon} & \downarrow p \\ [t - \epsilon(t), t + \epsilon(t)] \times D^k & \longrightarrow & B \end{array}$$

Moreover, since the map $f \circ h'_t : \{t\} \times D^k \rightarrow V$ is fiberwise homotopic to the restriction $h|_{\{t\} \times D^k}$ relative to $\{t\} \times \partial D^k$, it follows that for each $s \in [t - \epsilon(t), t + \epsilon(t)]$, the map

$$\{s\} \times D^k \subseteq [t - \epsilon(t), t + \epsilon(t)] \times D^k \xrightarrow{h'_{t,\epsilon}} E \xrightarrow{f} V$$

is fiberwise homotopic to the corresponding restriction of h relative to $\{s\} \times \partial D^k$. This uses that q is a Serre fibration.

By the Lebesgue lemma, there is $N > 0$ and maps

$$h'_i : [i/N, (i+1)/N] \times D^k \rightarrow E$$

for $i = 0, \dots, N-1$, such that:

- (i) h'_i and h'_{i+1} agree on $\{(i+1)/N\} \times \partial D^k$,
- (ii) $p \circ h'_i$ agrees with b on $[i/N, (i+1)/N]$,
- (iii) the restrictions of $f \circ h'_i$ and $f \circ h'_{i+1}$ to $\{(i+1)/N\} \times D^k$ are homotopic relative to $\{(i+1)/N\} \times \partial D^k$, by a fiberwise homotopy over B , to the restriction of h .

The connectivity assumption and (iii) imply that the restrictions of

$$h'_i, h'_{i+1} : \{(i+1)/N\} \times D^k \rightarrow p^{-1}(b(\frac{i+1}{N}))$$

are homotopic relative $\{(i+1)/N\} \times \partial D^k$. Thus, inductively, we obtain diagrams as follows, $i \geq 0$,

$$\begin{array}{ccc} \{\frac{i+1}{N}\} \times D^k \cup [\frac{i+1}{N}, \frac{i+2}{N}] \times \partial D^k & \xrightarrow{\quad} & E \\ \downarrow & \searrow^{h'_{i+1}} & \downarrow p \\ [\frac{i+1}{N}, \frac{i+2}{N}] \times D^k & \xrightarrow{\quad} & [\frac{i+1}{N}, \frac{i+2}{N}] \xrightarrow{b} B \end{array}$$

where the top map is defined by h'_i and u and the upper triangle commutes up to fiberwise homotopy. Then Lemma 2.2 shows that there is a map

$$\bar{h}_{i+1} : [(i+1)/N, (i+2)/N] \times D^k \rightarrow E$$

so that the diagram commutes strictly. This way, we inductively adjust the lifts h'_i so that they agree at the endpoints and obtain an extension of u ,

$$\bar{h} : [0, 1] \times D^k \rightarrow E.$$

The map $f \circ \bar{h}$ is then homotopic to h fiberwise over B and relative to $[0, 1] \times \partial D^k$ because their restrictions to $\{0\} \times D^k$ are homotopic, by construction, and q is a Serre fibration. \square

Proof of Theorem 1.3. We show that every diagram

$$(1) \quad \begin{array}{ccc} \{0\} \times I^k & \xrightarrow{u} & E \\ \downarrow j & & \downarrow p \\ [0, 1] \times I^k & \xrightarrow{v} & B \end{array}$$

admits a diagonal filler for all $k \leq n$.

Since $q : V \rightarrow B$ is a Serre fibration, there is a map g such that the following diagram commutes:

$$\begin{array}{ccc} \{0\} \times I^k & \xrightarrow{f \circ u} & V \\ \downarrow j & \nearrow g & \downarrow q \\ [0, 1] \times I^k & \xrightarrow{v} & B \end{array}$$

By Lemma 2.3, the map

$$(f^{I^{k+1}})_v : (p^{I^{k+1}})^{-1}(v) \rightarrow (q^{I^{k+1}})^{-1}(v)$$

is 0-connected for $k \leq n$. Therefore there is a map $h : [0, 1] \times I^k \rightarrow E$ so that the lower triangle of (1) commutes and $f \circ h$ belongs to the same path component of $(q^{I^{k+1}})^{-1}(v)$ as g does.

We claim that u and $h \circ j$ are fiberwise homotopic over B . By Lemma 2.3, the map

$$(f^{I^k})_{v \circ j} : (p^{I^k})^{-1}(v \circ j) \rightarrow (q^{I^k})^{-1}(v \circ j)$$

is 1-connected for $k \leq n$. The maps $f \circ u$ and $f \circ (h \circ j)$ are in the same path component of $(q^{I^k})^{-1}(v \circ j)$ and hence so are the maps u and $h \circ j$. Therefore, using Lemma 2.2, it follows that a diagonal filler to the lifting problem (1) exists, hence p is a Serre n -fibration.

Since p is a Serre n -fibration, the map $p^{-1}(b) \rightarrow \text{hofib}_b(p)$ is n -connected. Then the lower map between homotopy fibers in the diagram,

$$\begin{array}{ccc} p^{-1}(b) & \xrightarrow{(n+1)\text{-conn}} & q^{-1}(b) \\ \downarrow n\text{-conn} & & \downarrow \sim \\ \text{hofib}_b(p) & \longrightarrow & \text{hofib}_b(q) \end{array}$$

is $(n+1)$ -connected. By the long exact sequence of homotopy groups, it follows that f is $(n+1)$ -connected. \square

Remark 2.4. It would be interesting to investigate partial converses of Theorem 1.3. The problem is as follows: given a Serre microfibration $p : E \rightarrow B$, a Serre fibration $q : V \rightarrow B$ and a map $f : E \rightarrow V$ over B which is a weak homotopy (or n -)equivalence, when are the maps $f_b : p^{-1}(b) \rightarrow q^{-1}(b)$ also weak homotopy equivalences? If p is a Serre n -fibration and f is n -connected, then it is easy to see that f_b is n -connected for all $b \in B$, however, the converse claim is not true in general.

3. APPLICATIONS

3.1. Relative configuration spaces. In [8], we considered configuration spaces associated with maps as an attempt to deal with the lack of functoriality. We recall that given a space X , the space of (ordered) configurations of n points in X is defined to be the subspace $F_n(X) \subset X^n$ consisting of the n -tuples of pairwise distinct points. If X is a topological manifold, then it is well-known that the projection onto the first n -coordinates, $\pi_n : F_{n+1}(X) \rightarrow F_n(X)$, is a fiber bundle.

Given a map $f : X \rightarrow Y$, the relative (ordered) configuration space of n points $F_n(f)$ is the subspace of $F_n(X)$ which consists of the configurations in X whose

images in Y are n -tuples of pairwise distinct points. There is an obvious zigzag of maps

$$F_n(X) \leftarrow F_n(f) \xrightarrow{f_*} F_n(Y).$$

If Y is a Hausdorff space, then $F_n(f) \subset F_n(X)$ is open. If in addition X is a topological manifold, then the projection

$$F_{n+1}(f) \rightarrow F_n(f)$$

is a Serre microfibration. Indeed, $F_{n+1}(f)$ is then an open subspace of the following pullback fibration:

$$\begin{array}{ccc} F_{n,1}(f, X) & \longrightarrow & F_{n+1}(X) \\ \downarrow & & \downarrow \pi_n \\ F_n(f) & \longrightarrow & F_n(X) \end{array}$$

We recall the definition of cellular and cell-like maps, see e.g., [4]. A compact subset $K \subset M$ of a topological d -manifold is *cellular* if there are d -dimensional balls $B_i \subset M$, $i \geq 1$, with $B_{i+1} \subset \text{int}(B_i)$ and

$$K = \bigcap_{i=1}^{\infty} B_i.$$

It follows that, for $x \in K$, the spaces $M - \{x\}$ and $M - K$ are homeomorphic (see [2]) and the inclusion

$$M - K \longrightarrow M - \{x\}$$

is a homotopy equivalence. A map $f : M \rightarrow Y$ is *cellular* if $f^{-1}(y) \subset M$ is cellular for all $y \in Y$. A space K is *cell-like* if there is a topological manifold M and an embedding $\iota : K \hookrightarrow M$ such that $\iota(K)$ is cellular in M [4]. There exist embeddings of cell-like spaces that are not cellular, that is, cellularity depends on the embedding - see [6] for a general cellularity criterion. For finite-dimensional compact metric spaces, being cell-like is equivalent to having trivial shape [5]. A map $f : X \rightarrow Y$ is *cell-like* if $f^{-1}(y)$ is cell-like for all $y \in Y$. Proper cell-like maps between ENRs are hereditary proper homotopy equivalences [4, Theorem 1.2].

We obtain the following result which generalizes [8, Theorem 4.5].

Proposition 3.1. *Let M be a topological d -manifold (without boundary), Y an ENR and $f : M \rightarrow Y$ a proper cellular map. Then the maps $F_n(f) \rightarrow F_n(M)$ and $f_* : F_n(f) \rightarrow F_n(Y)$ are homotopy equivalences for all $n \geq 1$.*

Proof. We show inductively that $F_n(f) \rightarrow F_n(M)$ is a weak homotopy equivalence. Consider the following diagram

$$\begin{array}{ccccc} F_n(f) & \longrightarrow & F_{n,1}(f, M) & \xrightarrow{\sim} & F_n(M) \\ & \searrow p & \downarrow q & & \downarrow \pi_n \\ & & F_{n-1}(f) & \xrightarrow{\sim} & F_{n-1}(M) \end{array}$$

where the square on the right is a pullback by definition. We know that the projection p is a Serre microfibration from the discussion above. For any $\underline{m} = (m_1, \dots, m_{n-1}) \in F_{n-1}(f)$, the map $p^{-1}(\underline{m}) \rightarrow q^{-1}(\underline{m})$ is given by the inclusion

$$(M - f^{-1}(f(\{m_1, \dots, m_{n-1}\}))) \rightarrow (M - \{m_1, \dots, m_{n-1}\}).$$

This is a weak homotopy equivalence because M is a manifold and f is cellular. Then it follows from Theorem 1.3 that the map $F_n(f) \rightarrow F_{n,1}(f, M)$ is also a weak homotopy equivalence and the result follows inductively.

The map $f_* : F_n(f) \rightarrow F_n(Y)$ is again proper and cell-like/cellular, and therefore a (proper) homotopy equivalence, see [8, Proposition 4.4]. \square

Remark 3.2. In the case where Y is also a topological d -manifold, Proposition 3.1 specializes to a different proof of [8, Theorem 4.5]. The statement in [8] only requires that f is proper and cell-like, but cell-like maps in this case are indeed cellular. This is a consequence of the cellularity criterion of McMillan [6] for dimension $d \neq 4$, and Repovš [9] for $d = 4$ (see also Lacher [4, Theorem 4.3]). In dimension 3, the cellularity of proper cell-like maps requires also the proof of the Poincaré conjecture. See [7] for a nice survey. In general, a proper, cell-like map $f : M \rightarrow Y$ where M is a topological manifold is called a cell-like resolution of Y . Cell-like resolutions play a crucial role in the recognition problem for topological manifolds.

3.2. Abstract transversality. We comment on the abstract transversality context of [1] and in particular, the results of [1, Theorem 3.7] and [3, Corollary 2.9]. First, as a consequence of Theorem 1.3, we have the following proposition.

Proposition 3.3. *Let B be a space, X_\bullet a simplicial space, and $p_\bullet : X_\bullet \rightarrow B$ a simplicial map which is a degreewise Serre fibration (B is regarded as a constant simplicial space). Let $U_\bullet \subseteq X_\bullet$ be a degreewise open subspace. Suppose that for all $k \geq 0$ and $b \in B$, the map*

$$U_k \cap p_k^{-1}(b) \rightarrow p_k^{-1}(b)$$

is $(n+1-k)$ -connected. Then the map $\|U_\bullet\| \rightarrow \|X_\bullet\|$ is $(n+1)$ -connected. (Here $\|\cdot\|$ denotes the fat realization of the simplicial space.)

Proof. By Theorem 1.3, the map $U_k \rightarrow X_k$ is $(n+1-k)$ -connected. Then it follows from well-known results about the fat realization (see, e.g., [3, Proposition 2.7]) that $\|U_\bullet\| \rightarrow \|X_\bullet\|$ is $(n+1)$ -connected. Note that the fat realization is the same as the geometric realization of the underlying semi-simplicial space. \square

The abstract transversality context concerns a setup similar to Proposition 3.3 but with more special and weaker assumptions. The setup involves a space B , a simplicial space F_\bullet , and an open subspace $U_\bullet \subseteq B \times F_\bullet$. The fiber F_\bullet at $b \in B$ of the projection

$$B \times F_\bullet \rightarrow B$$

is regarded as the space of potential structures on b . The subspace U_\bullet consists of those structures which are admissible or *transverse*. The question is whether a fiberwise transversality statement, that is, a statement that the map

$$\|U_{b,\bullet}\| := \|U_\bullet \cap (\{b\} \times F_\bullet)\| \rightarrow \|F_\bullet\|$$

is a weak equivalence for all $b \in B$, implies a global transversality statement saying that

$$\|U_\bullet\| \rightarrow \|B \times F_\bullet\|$$

is a weak equivalence. For concrete geometric examples of such questions, see [1] and [3]. Proposition 3.3 fails to fit directly in this context because it requires that the weak (or n -)equivalences are *degreewise*. The following proposition is a slight improvement of related results in [1] and [3] and concerns the case where the maps become fiberwise n -equivalences *after* geometric realization.

Proposition 3.4. *Let F_\bullet be a simplicial set and B a Hausdorff space. Let $p_\bullet : B \times F_\bullet \rightarrow B$ be the projection and $U_\bullet \subseteq B \times F_\bullet$ a degreewise open simplicial subspace. If*

$$\|U_{b,\bullet}\| \rightarrow \|F_\bullet\|$$

is an $(n+1)$ -equivalence for all $b \in B$, then so is $\|U_\bullet\| \rightarrow \|B \times F_\bullet\|$.

Proof. The map $\|U_\bullet\| \rightarrow B$ is a Serre microfibration [3, Proposition 2.8]. This uses that B is Hausdorff. Note that the fiber of this map at $b \in B$ is $\|U_{b,\bullet}\|$ (see [3, Corollary 2.9]). Then the result follows from Theorem 1.3. \square

Remark 3.5. Special cases of this appear in [3] and [1]. The authors of [3] considered the case where $\|U_{b,\bullet}\|$ are n -connected, see [3, Corollary 2.9]. On the other hand, in [1], it was shown under different point-set topological assumptions that if the maps $\|U_{b,\bullet}\| \rightarrow \|F_\bullet\|$ are weak equivalences, then so is the map $\|U_\bullet\| \rightarrow \|B \times F_\bullet\|$, cf. [1, 3.6, 3.7].

REFERENCES

- [1] Bökstedt, Marcel; Madsen, Ib, *The cobordism category and Waldhausen's K-theory*, An alpine expedition through algebraic topology, pp. 39-80, Contemp. Math. 617, Amer. Math. Soc., Providence, RI, 2014.
- [2] Christenson, C. O.; Osborne, R. P., *Pointlike subsets of a manifold*, Pacific J. Math. 24 (1968), 431-435.
- [3] Galatius, Søren; Randal-Williams, Oscar, *Homological stability for moduli spaces of high dimensional manifolds. I*, arXiv:1403.2334.
- [4] Lacher, R. C., *Cell-like mappings. I*, Pacific J. Math. 30 (1969), 717-731.
- [5] Lacher, R. C., *Cell-like spaces*, Proc. Amer. Math. Soc. 20 (1969), 598-602.
- [6] McMillan, D. R., Jr., *A criterion for cellularity in a manifold*, Ann. of Math. (2) 79 (1964) 327-337.
- [7] Mitchell, W. J. R.; Repovš, D., *The topology of cell-like mappings*, Conference on Differential Geometry and Topology (Sardinia, 1988), Rend. Sem. Fac. Sci. Univ. Cagliari 58 (1988), 265-300.
- [8] Raptis, George, *Some remarks on configuration spaces*, Proc. Amer. Math. Soc. 139 (2011), no. 5, 1879-1887.
- [9] Repovš, Dušan, *A criterion for cellularity in a topological 4-manifold*, Proc. Amer. Math. Soc. 100 (1987), no. 3, 564-566.
- [10] Weiss, Michael, *What does the classifying space of a category classify?*, Homology Homotopy Appl. 7 (2005), no. 1, 185-195.

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