SEMINAR ON STABLE HOMOTOPY THEORY
(SS17, WEDNESDAY 14-16, M 101)

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Seminar Schedule

1. The stable homotopy category

Talk 1 (26.04.2017): *Introduction to spectra.* The purpose of this talk is to introduce the category of spectra. We will follow the classical approach using CW spectra. Introduce the basic definitions and give many examples. This includes: (CW-)spectra, Ω-spectra, maps between spectra, homotopies, homotopy groups of spectra, and (weak) homotopy equivalences. Main references: [1, III.2], [9, Ch. 8].

Talk 2 (03.05.2017): *Homotopy theory of spectra.* The purpose of this talk is to discuss some important properties of the homotopy category of spectra. This includes the property that it is additive and that cofiber and fiber sequences coincide. Main references: [1, III.3], [9, Ch. 8]. Further reading: [4].

Talk 3 (10.05.2017): *Spectra and generalized (co)homology.* The purpose of this talk is to define the generalized homology and cohomology theories associated with a spectrum E and discuss some examples including ordinary singular (co)homology and bordism theories. The example of E-(co)homology with coefficients should also be discussed. Main references: [9, Ch. 8 and 9], [1, III.6], [3, 3.4].

Talk 4 (17.05.2017): *Brown representability.* This talk is concerned with the Brown representatibility theorem. State and prove the (unstable) Brown representability theorem. Deduce the stable version and explain how this establishes an equivalence between spectra and generalized cohomology theories. If time permits, the representability theorem of Adams could also be mentioned (without proof). Main references: [9, Ch. 9].

Talk 5 (24.05.2017): *Spanier-Whitehead duality.* The purpose of this talk is to explain the Spanier-Whitehead duality for finite spectra. Any required properties of the smash product of spectra will be stated without proof. Define the duality and deduce some its main properties. Then discuss the case of duality for a closed smooth manifold. If time permits, the representability of generalized homology theories could be sketched. Main references: [1, III.5], [9, pp. 321-335], [10, Ch. 7].

Talk 6 (31.05.2017): *The Atiyah-Hirzebruch spectral sequence.* The goal of this talk is to introduce/recall the machinery of spectral sequences and construct the Atiyah-Hirzebruch spectral sequence. Introduce the notion of an exact couple and explain the construction of the associated spectral sequence. Discuss the example of the spectral sequence associated with a filtered complex. Then construct the Atiyah-Hirzebruch spectral sequnce. Main references: [1, Ch. 7], [7, Ch. 7], [9, Ch. 15]. Further reading: [5].

Talk 7 (07.06.2017): *Modern foundations of stable homotopy theory: an introduction.* The purpose of this talk is to sketch the classical construction of the smash product of spectra and then discuss some modern approaches to stable homotopy theory where this construction has better properties and leads to richer stable homotopical structures.

2. Cohomology Operations And Applications

Talk 8 (14.06.2017): *Cohomology operations and Steenrod squares.* The purpose of this talk is to introduce the general notion of a (stable) cohomology operation and then discuss the properties of the Steenrod squares. Give the general definition of a (stable) cohomology operation and explain the connection with the cohomology of Eilenberg-MacLane spaces (spectra). Introduce the Steenrod squares axiomatically by stating their main defining properties. Deduce some relations between Steenrod squares. Main references: [7, Ch. 1 and 3], [2, 4.L]. Further reading: [9, Ch. 18], [11, V.8, VIII. 4-6].

Talk 9 (21.06.2017): *Construction of the Steenrod squares.* The purpose of this talk is to present the classical construction of the Steenrod squares. Define the Steenrod squares and sketch the verification of the some of their defining properties. State the calculation of the cohomology of Eilenberg-MacLane spaces/spectra (without proof). Main references: [7, Ch. 2], [2, 4.L], [9, Ch. 18]. Further reading: [3, 3.5], [11, VIII].

Talk 10 (28.06.2017): *The Hopf invariant one problem.* This talk is devoted to an application of the Steenrod squares to the Hopf invariant one problem. Main references: [7, Ch. 4], [2, 4.L].

Talk 11 (05.07.2017): *The Steenrod algebra and its modules.* The purpose of this talk is to define the Steenrod algebra and study its properties. Define the Steenrod algebra of stable cohomology operations and deduce a linear basis and a set of generators in terms of Steenrod squares. Explain the Hopf algebra structure of the Steenrod algebra. Define modules over the Steenrod algebra and give some main examples. If time permits, some structure theorems about modules could be presented. Main references: [7, Ch. 6], [6], [8, Ch. IV].

Talk 12 (12.07.2017): *Calculation of the unoriented cobordism ring.* The goal of this talk is to calculate the unoriented cobordism ring $\pi_* MO$. Identify the cohomology of MO as a module over the Steenrod algebra. Deduce the calculation of its homotopy groups. Explain the comparison between unoriented bordism theory and ordinary homology theory. Main references: [8, Ch. VI]. Further reading: [3].

Talk 13 (19.07.2017): *The homotopy invariance of the Stiefel-Whitney classes.* The goal of this talk is to show the homotopy invariance of the Stiefel-Whitney classes of a closed smooth manifold. Show how the Stiefel-Whitney classes can be defined in terms of Steenrod squares. Define the Wu classes of a closed manifold. Show the relationship between the Stiefel-Whitney classes and the Wu classes which then implies the main result. References: [8, Ch. VI].

Talk 14 (26.07.2017): *The Adams spectral sequence: an introduction.* An outline of the construction of the Adams spectral sequence will be presented and some of its applications to the calculation of cobordism groups will be discussed. Main references: [1], [3], [5], [9].

References

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[6] Milnor, J. W. and Moore, J. C., *On the structure of Hopf algebras,* Ann. of Math. (2) 81, 1965, 211–264.

[7] Mosher, R. E. and Tangora, M. C., *Cohomology operations and applications in homotopy theory.* Harper & Row, Publishers, New York-London, 1968.

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[9] Switzer, R. M., *Algebraic topology - homotopy and homology.* Reprint of the 1975 original. Classics in Mathematics. Springer-Verlag, Berlin, 2002.

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