SEMINAR ON STABLE HOMOTOPY THEORY II (WS17/18, WEDNESDAY 16-18, SFB SEMINAR-ROOM)

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Seminar Description

The seminar will focus on calculations of unstable and stable homotopy groups. For that purpose, we will introduce and study two important theoretical tools: first, the Serre spectral sequence and Serre class theory, and secondly, the Adams spectral sequence.

Seminar Schedule

1. The Serre spectral sequence and its applications

Talk 1 (18.10.2017 – Gesina Schwalbe): *Overview of some classical results.* Recall the classical theorems of Freudenthal and Hurewicz and some elementary calculations of homotopy groups of spheres. Recall the definition of the stable homotopy groups of spheres and explain the ring structure. Show that the Hopf maps are stably essential, i.e. not stably null-homotopic, using Steenrod operations. Use the Adem relations to show that some products of Hopf maps are also stably essential – explain the idea carefully in one example and quickly discuss a second example. Main references: [4, esp. 4.L], [11, Chapter 1], [10].

Talk 2 (25.10.2017 – Julian Seipel): *Construction of the Serre spectral sequence.* Explain the construction of the Serre spectral sequence for singular (co)homology following [3]. This means that you should explain first the 2 spectral sequences one obtains from a double complex. You may state the version with twisted coefficients and explain the proof only in the untwisted case. Discuss some of its immediate consequences such as the Serre exact sequence. Main references: [3], [5, pp. 526-532, pp. 542-543], [9, 5.1-5.2], [10, Chapter 8].

Talk 3 (08.11.2017 – Jonathan Glöckle): *Applications of the Serre spectral sequence. I.* Discuss the application of the Serre spectral sequence to the path space fibration. Introduce Serre classes and explain some applications of the Serre spectral sequence for specific examples of Serre classes. Main references: [5, pp. 532-536], [9, 5.2], [10, Chapters 8 and 10].

Talk 4 (15.11.2017 – Markus Land): *Further properties of the Serre spectral sequence.* Explain the multiplicative properties of the Serre spectral sequence – an account based upon Dress' construction can be found in [6, chapter 7]. In particular, first recall what it means to have a multiplicative filtration on a double complex and how this yields a multiplicative spectral sequence. Then explain that the Dress construction gives a multiplicative filtered complex. Discuss some more examples, computations, and the transgression. Main references: [5, pp. 540-551], [9, Chapter 6], [10, Chapter 11].

Talk 5 (22.11.2017 – Luigi Caputi): *Applications of the Serre spectral sequence. II.* Calculate the cohomology of $K(\mathbb{Z}/2\mathbb{Z}, n)$ and other Eilenberg-MacLane spaces using the Serre spectral sequence. Main references: [5, pp. 562-569], [10, Chapter 9].

Talk 6 (29.11.2017 – Han-Ung Kufner): *Calculating homotopy groups. The rational case.* Explain Serre's method for the calculation of homotopy groups of spheres and use it to calculate $\pi_5(S^3)$ as an example. Then calculate the rational homotopy groups of spheres. Conclude that $\pi_*^s \otimes \mathbb{Q} \cong \mathbb{Q}$. Main references: [5, pp. 551-552, pp. 557-559], [10].

Talk 7 (06.12.2017): *Calculating homotopy groups of spheres. p-torsion elements.* Identify some p-torsion elements in the homotopy groups of spheres and in particular show that the first p-torsion in π_*^s appears in degree $2p-3$. Later we will revisit and expand this fact by means of the Adams spectral sequence. More computations of homotopy groups of spheres. Main references: [5, pp. 560-562, pp. 573-578], [9, Chapter 6], [10, Chapter 12].

2. The Adams spectral sequence and its applications

Talk 8 (13.12.2017 – Georgios Raptis): *Background from homological algebra.* Briefly recall the definition of an abelian category and its derived category. Discuss Ext-groups in this context, see for instance [13, chapter 10.7]. Then recall the Steenrod algebra as the algebra of graded endomorphisms of $\mathrm{H}\mathbb{F}_2$ and that the cohomology of spectra gives examples of modules over the Steenrod algebra. Explain examples of extensions of modules over the Steenrod algebra – e.g. the ones coming from elements of $\pi_n(S^k)$ as in [5, pp. 580]. Main reference: [1].

Talk 9 (20.12.2017 – Gesina Schwalbe): *Overview of spectra and the stable homotopy category.* Recall the general properties of the stable homotopy category. As a warm-up discuss the p-localisation of spectra. Then continue to discuss p-completions, and describe the homotopy groups of the p-completion of a spectrum X in terms of the homotopy groups of X, i.e. discuss the derived p-completion of abelian groups. Show that derived and ordinary p-completion agree on finitely generated abelian groups but not in general. If time permits, show that $\mathbb{S}/2$ does not admit a multiplication as an example that not all algebraic processes can be transported to "higher algebra" – the theory of ring spectra. Main references: *Stable homotopy theory I*, [2], [8].

Talk 10 (09.01.2017 – Julian Seipel): *Construction of the Adams spectral sequence. I.* Define Adams resolutions and explain the construction of the Adams spectral sequence. Determine the E_2-page of the Adams spectral sequence as Ext-groups over the Steenrod algebra. If you have time discuss the toy examples where the cohomology of the spectrum we investigate is $\mathcal{A}_*$ and $\mathcal{A}_*/\beta$. Main references: [2], [5, pp. 594-599], [7, Chapter 3.6], [9, Chapter 9], [11].

Talk 11 (16.01.2017 – Jonathan Glöckle): *Construction of the Adams spectral sequence. II.* Continuation of the construction of the Adams spectral sequence. Explain that the Adams filtration one obtains on homotopy groups is complete and Hausdorff and that a map of spectra induces a filtration preserving map on homotopy groups. Discuss convergence of the spectral sequence and minimal resolutions. Main references: [2], [9], [11].

Talk 12 (23.01.2018): *Applications of the Adams spectral sequence. I.* Discuss the Adams spectral sequence of the sphere and focus on the prime 2: Explain what it means that an element in an Ext-group represents an element in the stable stem. Then discuss the elements $h_i \in \mathrm{Ext}^1_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2[2^i])$ on the 1-line of the spectral sequence, relating them to the Hopf maps, and try to explain the differential $d_2(h_i) = h_0 h_{i-1}^2 \neq 0$ for $i \geq 4$. If time permits, prove that the first two p-torsion groups appear in degrees $2p-3$ and $4p-5$ if p is an odd prime. Main references: [5, pp. 599-604], [1], [9, Chapter 9].

Talk 13+14 (30.01.2018 + 07.02.2018): *Applications of the Adams spectral sequence. II.* Discuss applications of the Adams spectral sequence to cobordism theories, more precisely calculate the homotopy groups of MO and MU. Main references: [7], [9, Chapter 9], [12, Chapter 20], [11].

References

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[5] Hatcher, A., Chapter on *Spectral sequences*, available online https://www.math.cornell.edu/ hatcher/AT/ATch5.pdf.

[6] Hebestreit, F., Krause A. and Nikolaus, T. *Spectral sequences* lecture notes, available online at http://www.math.uni-bonn.de/people/fhebestreit/ATII

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