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Summer Semester 2021

Universität Regensburg
Fakultät für Mathematik

Seminar: Topics in Homotopy Theory
Schedule & Overview of Talks
(Wednesday 16–18, online)

Part 1: The Serre spectral sequence and its applications

Prerequisites. *Essential:* Basic homotopy theory of topological spaces (e.g., singular and cellular (co)homology, fibrations, definition and properties of homotopy groups, Eilenberg-MacLane spaces, etc.) and homological algebra (e.g., (co)chain complexes, (co)homology, long exact sequences, etc.). *Desirable:* It will also be useful to be familiar with singular cohomology with twisted coefficients and generalized (co)homology theories.

Preliminary Reading. For general background material about the homotopy theory of topological spaces and homological algebra, see [5, 11] and [8, Ch. 4]. *Recommended:* Flicking through the informal introduction in [8, Ch. 1] is highly recommended.

Talk 1 (14.04.2021): *Exact couples and spectral sequences: Theory.* Introduce the notion of an exact couple and explain the construction of the associated spectral sequence. Discuss the example of the spectral sequence associated with a filtered complex. Mention some simple examples. Main references: [6, pp. 520–526], [9, Ch. 7], [11, Ch. 15], [8, Ch. 1 and 2.1–2.2].

Talk 2 (21.04.2021): *Exact couples and spectral sequences: First examples.* Discuss some examples of spectral sequences and their applications. Possible examples are: the spectral sequence(s) of a double complex, the Bockstein exact couple, the Atiyah-Hirzebruch spectral sequnce, the homology spectral sequence of a simplicial space. Main references: [9, Ch. 7], [11, Ch. 15], [1, Ch. 7], [4, Sec. 12], [8, 2.4 and 11.2].

Talk 3 (28.04.2021): *The Serre spectral sequence.* Explain the construction of the Serre spectral sequence for singular (co)homology. It may be a good idea to explain the proof only in the untwisted case. Discuss some elementary examples of the Serre spectral sequence and some of its immediate consequences (e.g., the Serre exact sequence, the Gysin sequence, etc.). Main references: [6, pp. 526–532, 536–540, 542–543], [8, 5.1–5.2], [9, Ch. 8].

Talk 4 (05.05.2021): *Applications of the Serre spectral sequence.* Discuss the Serre spectral sequence in the case of the path space fibration. Introduce Serre classes and explain some applications of the Serre spectral sequence for specific examples of Serre classes. Main references: [6, pp. 532–536], [8, 5.2], [9, Ch. 8 and 10].

Talk 5 (12.05.2021): *Further properties of the Serre spectral sequence.* Sketch the proof of the multiplicative properties of the Serre spectral sequence. Discuss some more examples, computations, and the transgression. Main references: [6, pp. 540–551], [8, Ch. 5.1–5.2, 6.2, 6.4], [9, Ch. 8 and 11].

Talk 6 (19.05.2021): *Calculating homotopy groups.* State the calculation of the cohomology of Eilenberg-MacLane spaces (without proof) and explain Serre's method for the calculation of homotopy groups. Compute some examples including the rational homotopy groups of spheres. Main references: [6, pp. 551–552, 557–577] , [9, Ch. 9 and 12], [8, Ch. 6]

Further Reading. There are many other important examples of (co)homology and homotopy groups which can be computed using the Serre spectral sequence; see [8, 5.2 and Ch. 6], [9, Ch. 12], [6]. Moreover, there are various other spectral sequences which are very useful in homotopy theory (e.g. the Adams spectral sequence); see [8], [11], [1].

Part 2: Localizations of Spaces

Prerequisites. *Essential:* Familiarity with basic notions from the homotopy theory of topological spaces (esp. fibrations, homotopy fibers, Eilenberg–MacLane spaces, etc.); (algebraic) localization of abelian groups and its properties. *Desirable:* It would be useful to review the action of the fundamental group on the homotopy groups, as this will be important in order to understand the subtleties of the non-simply-connected case.

Preliminary Reading. *Essential:* For an overview of the properties of localizations of abelian groups, see [7, 5.1], [10, Sec. 3–4]. A casual preliminary reading of parts of [7, Ch. 7] might also be useful. *Recommended:* For the general notion of localization from a category–theoretic viewpoint, see [4, Sec. 14], [10, Sec. 1].

Talk 7 (26.05.2021): *Postnikov towers and obstruction theory.* Introduce the construction of the Postnikov tower and its properties (for simply–connected, simple, and nilpotent spaces). Explain the use of this construction in obstruction theory. If time permits, discuss also briefly the Moore–Postnikov tower of a map. Main references: [5, pp. 410–419], [7, Ch. 3], [9, Ch. 13].

Talk 8 (02.06.2021): *Localizations of spaces: Theory.* Introduce the notion of the R-localization (or p-localization) of a topological space ($R \subset \mathbb{Q}$) and explain the existence of these localizations for nice spaces (= nilpotent, simple, or simply-connected). Discuss some of the properties of p-localization (or at a set of primes) and its equivalent characterizations. If time permits, the alternative construction in [10, Sec. 1–2 and 5], [7, 6.5] could be briefly mentioned. Main references: [7, 5.2–5.3 and 6.1], [4, Sec. 15–16], [6, pp. 553–557], [10].

Talk 9 (09.06.2021): *Localizations of spaces: Examples and Applications.* Discuss further characterizations of local spaces and localizations. Describe the effect of R-localization on various standard homotopy–theoretic constructions (e.g. homotopy (co)fibers) – without complete proofs. Then discuss some examples and applications (e.g., $S^n_{\mathbb{Q}}$ and rational H-spaces). Main references: [7, Ch. 6 and 9.1], [4, Sec. 14–16], [6, pp. 557–562].

Talk 10 (16.06.2021): *Fracture theorems for localization.* It might be helpful to begin with the corresponding theorems for (nilpotent or abelian) groups [7, 7.1–7.2]. Then state and prove the fracture theorems for the localization of (simply-connected) spaces/Sullivan's arithmetic square for the p-localizations of a (simply-connected) space. Main references: [7, 8.1–8.4], [4, Sec. 17].

Further Reading. The closely related construction of p-completion (of groups or spaces) is a natural continuation of this theory. See [4, Sec. 18–21], [7, Part 3], [10, Sec. 6–9].

Part 3: Homotopy Colimits

Prerequisites. *Essential:* Familiarity with basic notions from the homotopy theory of topological spaces (e.g., cofibrations, fibrations, mapping cone/homotopy cofiber, homotopy fiber, etc.); familiarity with the language of category theory (e.g., categories, functors, adjoint functors, limits and colimits, etc.); the construction of (co)limits in the category of topological spaces. *Desirable:* It will be useful to know some additional special cases of homotopy (co)limit constructions (e.g., double mapping cylinders/homotopy pushouts, homotopy pullbacks, etc.).

Preliminary Reading. *Essential:* For an informal introduction to homotopy (co)limits, see [2, 1.2]. For a review of (co)fibrations and homotopy (co)fibers in the homotopy theory of topological spaces, see [7, Ch. 1]; these topics are treated in most of the standard textbooks on homotopy theory (see also [5, 11]). *Recommended:* A quick look at the introductory chapters of [3] is highly recommended (esp. [3, Ch. I and V]). For the classical constructions of homotopy pushouts and homotopy pullbacks and their properties, see [7, 2.1–2.2].

Talk 11 (23.06.2021): *Construction of homotopy colimits of spaces.* Review some basic facts about the homotopy theory of simplicial spaces and introduce the construction of the homotopy colimit. Explain the proof of the homotopy invariance of the homotopy colimit and discuss some examples. If time permits, discuss briefly the dual construction of the homotopy limit. Main references: [2, 1.3–1.4], [4, Sec. 4–6], [7, Ch. 2].

Talk 12 (30.06.2021): *Homotopy colimits as derived functors.* Introduce the notion of a derived functor and prove that the homotopy colimit constructions models the derived functor of the colimit. If time permits, state also the corresponding results for the homotopy limit. Main references: [4, Sec. 2–3, 7–8], [2, 2.9, 2.11]. See also [3].

Talk 13 (07.07.2021): *Spectral sequences for homotopy colimits.* Explain the construction of the homology spectral sequence of a homotopy colimit and discuss some examples. State the corresponding spectral sequence for the homotopy limit. Main references: [2, Sec. 18], [4, Sec. 12–13]. See also [8].

Talk 14 (14.07.2021): *Homotopy inverse limits and* $\lim^1$. Discuss the special case of homotopy inverse limits. Introduce $\lim^1$, discuss its algebraic properties, and explain its connection with the identification of phantom maps. Prove the Milnor short exact sequence for the homotopy groups of a homotopy inverse limit. Main references: [7, Ch. 2].

Further Reading. For further interesting examples of homotopy colimits, see [4, Sec. 9–10], [2, Sec. 22]. For the general theory of homotopy (co)limits in homotopical algebra (using model categories), see [2, Part 3] and [3].

References

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[2] Dugger, D., *A primer on homotopy colimits.* `https://pages.uoregon.edu/ddugger/hocolim.pdf`

[3] W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, J. H. Smith, *Homotopy limit functors on model categories and homotopical categories.* Mathematical Surveys and Monographs Vol. 113, American Mathematical Society, Providence, RI, 2004.

[4] Grodal, J. and Haugseng, R., *Categories and Topology.* Lecture notes (2017). `https://folk.ntnu.no/runegha/notes.html`

[5] Hatcher, A., *Algebraic topology.* Cambridge University Press. Cambridge, 2002.

[6] Hatcher, A., Chapter on *Spectral sequences.* available online `https://www.math.cornell.edu/~hatcher/AT/ATch5.pdf`

[7] May, J. P. and Ponto, K., *More Concise Algebraic Topology.* University of Chicago Press (2012).

[8] McCleary, J., *A user's guide to spectral sequences.* Cambridge Studies in Advanced Mathematics Vol. 58. Cambridge University Press, Cambridge, 2001.

[9] Mosher, R. E. and Tangora, M. C., *Cohomology operations and applications in homotopy theory.* Harper & Row, Publishers, New York-London, 1968.

[10] Joseph A. Neisendorfer, *A Quick Trip through Localization.* Available online at: `https://people.math.rochester.edu/faculty/jnei/localization.pdf`

[11] Switzer, R. M., *Algebraic topology - homotopy and homology.* Reprint of the 1975 original. Classics in Mathematics. Springer-Verlag, Berlin, 2002.