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Summer Semester 2022

Universität Regensburg
Fakultät für Mathematik

Seminar: Homotopical Algebra – Model Categories
Schedule & Overview of Talks
(Wednesday 16–18, H31/online)

Description. This seminar will be an introduction to the theory and applications of model categories. The theory of model categories establishes an axiomatic framework for homotopy theory which has been very successful in analyzing homotopical phenomena and generalizing the methods of homological algebra (derived functors, etc.) to more general (non-additive) categories. The classical homotopy theories of topological spaces and chain complexes fit nicely in this framework and provide some of the main examples. Moreover, the theory of model categories provides a general framework in which different homotopy theories can be compared.

This seminar should be of interest to anyone interested in algebraic topology, homological algebra and/or (higher) categorical methods in topology and algebra. It is recommended to attend the seminar in parallel with *Algebraic Topology II*.

The main references for the seminar are the books by Quillen [Qu], Hovey [Ho], Hirschhorn [Hi], and the article by Dwyer–Spalinski [DS].

Date and place. Wednesday 16–18 (Summer Semester 2022) in H31 (or online). The preliminary meeting for the seminar will be on Wednesday, 9. February 2022, 16–18, in M311.

Prerequisites. *Essential*: Basic notions from category theory (e.g., categories, functors, natural transformations, adjoint functors, (co)limits, etc.). *Desirable*: Basic knowledge of algebraic topology and homological algebra.

Recommended Preliminary Reading (optional). We recommend flicking through the excellent articles by Dwyer–Spalinski [DS] and Goerss–Schemmerhorn [GS]. We also recommend [Ho2] for a brief description and historical survey of the theory of model categories.

Part 1: Basic Theory

Talk 1 (27.04.2022): *The definition of a model category and basic properties.* The axioms of a model category. The retract argument and its consequences. Ken Brown's lemma (with proof). Examples of model categories (without proof). Generalities about localizations of categories with weak equivalences. The main references are [Ho, pp. 2–8] and [DS, §3, §6]. See also [Hi, §7.1–7.2, 7.7].

Talk 2 (04.05.2022): *Homotopies and the homotopy category.* Cylinder objects and path objects. Left and right homotopies and their properties. Homotopy equivalences and comparison with weak equivalences. The homotopy category of a

model category. The main references are [Ho, pp. 8–13], [Hi, §7.3–7.5, 8.3], [DS, §4–5], [Qu, §I.1].

Talk 3 (11.05.2022): *Quillen adjunctions and derived functors.* Left and right Quillen functors and their properties. Total left and right derived functors and their universal properties. Quillen equivalences. Equivalent characterizations of Quillen equivalences. The main references are [Ho, §1.3], [Hi, §8.4-8.5]. See also [Qu, §I.4] and [DS, §9].

Further Reading (optional). See the monograph [DHKS] and [Ri] for further details on foundational aspects of homotopical algebra.

Part 2: Methods

Talk 4 (18.05.2022): *The small-object argument.* Notions of smallness of objects. Injectives, cell complexes, cofibrations and fibrations with respect to a class of morphisms. The small-object argument. The main references are [Ho, §2.1.1–2.1.2] and [Hi, §10.1–10.5].

Talk 5 (25.05.2022): *Cofibrantly generated model categories.* The definition of a cofibrantly generated model category and its basic properties. The recognition theorem for cofibrantly generated model categories. Application to the categories of diagrams in a cofibrantly generated model category and to the existence of homotopy colimits. The main references are [Ho, §2.1.3] and [Hi, §11.1–11.3, 11.5–11.6]. See also [DS, §10] and [GS, §3].

Further Reading (optional). There are interesting classes of cofibrantly generated model categories with additional special properties. You might want to read further about *combinatorial model categories* [Be, Ra] or *cellular model categories* [Hi, Chapter 12]. For more details about homotopy (co)limits, see [Du] and [DHKS].

Part 3: Examples

Talk 6 (01.06.2022): *Chain complexes.* The proof of the (projective) model structure on the category of chain complexes of R-modules. Examples of left/right Quillen functors and their derived functors. Other model category structures on the category of chain complexes and further generalizations (without proofs). The main reference is [Ho, §2.3]. See also [DS, §7], [GS, §1.1–1.4], [Qu, II.3].

Talk 7 (08.06.2022): *Topological spaces.* The proof of the standard (Kan–Quillen) model structure on the category of topological spaces. Examples of left/right Quillen functors and their derived functors. The Strøm model structure on topological spaces (to be mentioned without proof). Other related model categories (without proofs). The main reference is [Ho, §2.4]. See also [DS, 3.6, §8], [Qu, II.3].

Talk 8 (15.06.2022): *Simplicial sets.* Basic definitions and properties. (Trivial) (co)fibrations of simplicial sets. The geometric realization functor. Weak equivalences of simplicial sets. The main references are [Ho, §3.1–3.2] and [GJ, I.1–I.4]. See also [JT].

Talk 9 (22.06.2022): *Mapping spaces and (combinatorial) homotopy groups.* Mapping spaces and (co)fibrations. Homotopy groups of a fibrant (Kan) simplicial set. Long exact sequence of homotopy groups. The main references are [Ho, §3.3–3.4] and [GJ, I.5–I.7]. See also [JT].

Talk 10 (29.06.2022): *The model category of simplicial sets.* Fibrations and geometric realization. Outline of the proof of the model structure on the category of simplicial sets. The Quillen equivalence between simplicial sets and topological spaces. The main references are [Ho, §3.5–3.6] and [GJ, I.10–I.11]. See also [JT] and [Qu, II.3].

Further Reading (optional). See [GJ] for a detailed account of simplicial homotopy theory. There are several approaches to the model structure on simplicial sets. (It is interesting to compare [Ho, GJ] with [Qu, JT].) We also recommend the more advanced monograph [Ci] for an alternative and more general approach.

Part 4: Homotopical Algebra

Talk 11 (06.07.2022): *Simplicial model categories.* The axioms of a simplicial model category. Examples and non-examples. Mapping spaces and weak equivalences. Simplicial homotopy and left/right homotopy. Simplicial model categories of simplicial objects in a category. The main references are [GJ, II.2–II.5], [Qu, II.1–II.2, II.4] and [Hi, Chapter 9].

Talk 12 (13.07.2022): *Reedy model categories.* Reedy categories. The model category of diagrams over a Reedy category. Examples. The definition(s) of function spaces. The main references are [Ho, §5.1–5.2, 5.4], [Hi, Chapter 15], [GJ, VII.1–VII.2].

Talk 13 (20.07.2022): *Pointed and stable model categories.* The suspension and loop functors. Cofiber and fiber sequences and their properties. Stable model categories. Examples. The main references are [Ho, §6.1–6.3, 6.5, 7.1] and [Qu, I.2–I.3].

Talk 14 (27.07.2022): *Resolutions, homology, and cohomology.* The definition of homology and cohomology for model categories (of simplicial objects). Examples (e.g. André–Quillen homology). Resolution model categories (without proofs). The main references are [Qu, II.4–II.5], [GS, §4.3–4.4, 5.3].

References

[Be] T. Beke, *Sheafifiable homotopy model categories.* Math. Proc. Cambridge Philos. Soc. 129 (2000), no. 3, 447–475.

[Ci] D.-C. Cisinski, *Les préfaisceaux comme modèles des types d'homotopie.* Astérisque No. 308 (2006).

[Du] D. Dugger, *A primer on homotopy colimits.* Available at: `https://pages.uoregon.edu/ddugger/hocolim.pdf`

[DHKS] W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, J. H. Smith, *Homotopy limit functors on model categories and homotopical categories.* Mathematical Surveys and Monographs Vol. 113. American Mathematical Society, Providence, RI, 2004.

[DS] W. G. Dwyer, J. Spaliński, *Homotopy theories and model categories.* Handbook of algebraic topology, pp. 73–126, North-Holland, Amsterdam, 1995.

[GJ] P. G. Goerss, J. F. Jardine, *Simplicial homotopy theory.* Progress in Mathematics 174. Birkhäuser Verlag, Basel, 1999.

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[Hi] P. S. Hirschhorn, *Model categories and their localizations.* Mathematical Surveys and Monographs Vol. 99. American Mathematical Society, Providence, RI, 2003.

[Ho] M. Hovey, *Model Categories.* Mathematical Surveys and Monographs Vol. 63. American Mathematical Society, Providence, RI, 1999.

[Ho2] M. Hovey, *Quillen model categories.* J. K-Theory 11 (2013), no. 3, 469–478.

[JT] A. Joyal, M. Tierney, *Notes on simplicial homotopy theory.* Available online.

[Qu] D. Quillen, *Homotopical Algebra.* Lecture Notes in Mathematics Vol. 43. Springer, 1967.

[Ra] G. Raptis, *Notes on combinatorial model categories.* Available at: `https://graptismath.net/files/notes-comb-model-cat.pdf`

[Ri] E. Riehl, *Homotopical categories: From model categories to* $(\infty, 1)$*-categories.* Available at: `https://math.jhu.edu/~eriehl/homotopical.pdf`