IMAGES OF THE YONEDA LEMMA

A Rhizome[*]

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One by one objects are defined —
It quickens : clarity, outline of leaf

But now the stark dignity of
entrance — Still, the profound change
has come upon them : rooted they
grip down and begin to awaken

William Carlos Williams, *Spring and All*

Let Set denote the category of sets.

Let $\mathcal{C}$ be a small category and let $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ denote the category of presheaves on $\mathcal{C}$. For each object c in $\mathcal{C}$, there is an *associated representable presheaf*

$$\mathrm{y}_c \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set},\ x \mapsto \mathrm{Hom}_{\mathcal{C}}(x, c).$$

The representable presheaves determine the Yoneda functor

$$\mathrm{y} \colon \mathcal{C} \to \mathcal{P}(\mathcal{C}),\ c \mapsto \mathrm{y}_c.$$

For every morphism $(c \xrightarrow{u} c')$ in $\mathcal{C}$, the corresponding morphism in $\mathcal{P}(\mathcal{C})$

$$(\mathrm{y}(u) \colon \mathrm{y}_c \to \mathrm{y}_{c'})$$

is the natural transformation whose component at an object x of $\mathcal{C}$ is given by composition with u:

$$(x \to c) \mapsto (x \to c \xrightarrow{u} c').$$

Let $\mathcal{S}$ denote the ∞-category of spaces.

Let $\mathcal{C}$ be a small ∞-category and let

$$\mathrm{Map}_{\mathcal{C}}(-,-) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$$

be a functorial model for the mapping spaces in $\mathcal{C}$. We denote by $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ the ∞-category of presheaves on $\mathcal{C}$.

For each object c in $\mathcal{C}$, there is an *associated representable presheaf*

$$\mathrm{y}_c \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S},\ x \mapsto \mathrm{Map}_{\mathcal{C}}(x, c).$$

The functor $\mathrm{Map}_{\mathcal{C}}(-,-)$ determines by adjunction the Yoneda functor

$$\mathrm{y} \colon \mathcal{C} \to \mathcal{P}(\mathcal{C}),\ c \mapsto \mathrm{y}_c.$$

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[*]A botanical term used by Gilles Deleuze and Félix Guattari for a philosophical concept to describe *substantive multiplicities* – unities multiple in themselves, *assemblages*, and not multiplicities which are attributes of an assumed unity. For example: "*the rhizome connects any point to any other point, and its traits are not necessarily linked to traits of the same nature*" and "*[the rhizome] has neither beginning nor end, but always a middle from which it grows and which it overspills*". Also: "*unlike a structure, which is defined by a set of points and positions, with binary relations between the points and biunivocal relationships between the positions, the rhizome is made only of lines: lines of segmentarity and stratification as its dimensions, and the line of flight or deterritorialization as the maximum dimension after which the multiplicity undergoes metamorphosis, changes in nature.*"

This concept creates what Deleuze called *an image of thought* which is non-linear, horizontal, connectivist, and propagative; instead of components that are organized hierarchically or vertically as successive processes, it possesses entangled localities that are dynamically interlinked and enclose transformative directions which differentiate and regenerate the unity of the assemblage. This is an image of thought which resists foundationalist or structuralist (or *euclidean*?) analysis.

For every presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, there is a bijection

$$\lambda_c(F)\colon \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(\mathrm{y}_c, F) \xrightarrow{\cong} F(c)$$
$$(\theta\colon \mathrm{y}_c \to F) \mapsto \theta_c(\mathrm{id}_c) \in F(c),$$

which is natural in c and F (*Yoneda lemma*). For every morphism $(c \xrightarrow{u} c')$ in $\mathcal{C}$, we have

$$\mathrm{y}(u) = \big(\lambda_c(\mathrm{y}_{c'})\big)^{-1}(u).$$

Therefore, the Yoneda functor y is full and faithful (*Yoneda embedding*). In particular, an isomorphism $\mathrm{y}_c \cong \mathrm{y}_{c'}$ in $\mathcal{P}(\mathcal{C})$ yields canonically an isomorphism $c \cong c'$ in $\mathcal{C}$.

Let Cat denote the category of small categories and let e denote the terminal object in Cat.

The category $\mathrm{t}(\mathcal{C})$ is defined as follows: its objects are functors $(e \to \mathcal{C})$ and a morphism from $(e \xrightarrow{f} \mathcal{C})$ to $(e \xrightarrow{g} \mathcal{C})$ is given by a natural transformation $u\colon f \Rightarrow g$. The category $\mathrm{t}(\mathcal{C})$ is isomorphic to $\mathcal{C}$. The Yoneda embedding is identified with the functor

$$\mathcal{Y}_e\colon \mathrm{t}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$
$$(e \xrightarrow{f} \mathcal{C}) \mapsto (x \mapsto f_{x/}).$$

The *Grothendieck construction* (or category of elements) defines a functor

$$\textstyle\int\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Cat}_{/\mathcal{C}}$$
$$(F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}) \mapsto (p_F\colon \textstyle\int F \to \mathcal{C}).$$

This functor is full and faithful and its essential image is given by the cartesian fibrations over $\mathcal{C}$ whose fibers are discrete categories. Then the composite functor

$$\mathcal{C} \cong \mathrm{t}(\mathcal{C}) \xrightarrow{\mathcal{Y}_e} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \xrightarrow{\int} \mathrm{Cat}_{/\mathcal{C}}$$

is also full and faithful. The value of this composite functor at an object c of $\mathcal{C}$ is the cartesian fibration $\mathcal{C}_{/c} \to \mathcal{C}$.

For every presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, there is an equivalence

$$\lambda_c(F)\colon \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(\mathrm{y}_c, F) \xrightarrow{\simeq} F(c)$$
$$(\theta\colon \mathrm{y}_c \to F) \mapsto \big(\theta_c(\mathrm{id}_c)\colon * \to F(c)\big),$$

which is natural in c and F (*∞-categorical Yoneda lemma*). For every morphism $(c \xrightarrow{u} c')$ in $\mathcal{C}$, we have

$$\lambda_c(\mathrm{y}_{c'})\colon \mathrm{y}(u) \mapsto u.$$

Therefore, the Yoneda functor y is full and faithful (*∞-categorical Yoneda embedding*). The functors $\mathcal{P}(\mathcal{C}) \to \mathcal{S}$ corepresented by representable presheaves preserve small (co)limits and jointly detect the equivalences in $\mathcal{P}(\mathcal{C})$.

The category $\mathrm{t}(\mathcal{C})$ is a full subcategory of a category $\mathrm{T}(\mathcal{C})$. An object of $\mathrm{T}(\mathcal{C})$ is a functor $(\mathcal{D} \xrightarrow{f} \mathcal{C})$ in Cat. A morphism from $(\mathcal{D} \xrightarrow{f} \mathcal{C})$ to $(\mathcal{D}' \xrightarrow{g} \mathcal{C})$ is given by a functor $(\mathcal{D} \xrightarrow{q} \mathcal{D}')$ together with a natural transformation $(\theta\colon f \Rightarrow g \circ q)$.

The (Yoneda) functor $\mathcal{Y}_e$ extends to a functor

$$\mathcal{Y}\colon \mathrm{T}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat})$$
$$(\mathcal{D} \xrightarrow{f} \mathcal{C}) \mapsto (x \mapsto f_{x/}).$$

For every morphism $(q,\theta)\colon (\mathcal{D} \xrightarrow{f} \mathcal{C}) \to (\mathcal{D}' \xrightarrow{g} \mathcal{C})$ in $\mathrm{T}(\mathcal{C})$, the corresponding morphism

$$\mathcal{Y}(q,\theta)\colon \mathcal{Y}(f) \to \mathcal{Y}(g)$$

is the natural transformation whose component at an object x of $\mathcal{C}$ is the functor

$$\mathcal{Y}(q,\theta)_x\colon f_{x/} \to g_{x/}$$

defined on objects by

$$(d, x \to f(d)) \mapsto \big(q(d), x \to f(d) \xrightarrow{\theta_d} g(q(d))\big).$$

The functor $\mathcal{Y}$ is full and faithful (*Cat-version of the Yoneda embedding*).

Both $\mathrm{T}(\mathcal{C})$ and $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat})$ can be regarded as 2-categories, using that Cat is also a 2-category. A 2-morphism from (q,θ) to (q',θ') in $\mathrm{T}(\mathcal{C})$ is given by a natural transformation $\alpha\colon q \Rightarrow q'$ which is compatible with θ and θ'. The 2-morphisms in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat})$ are given by modifications between natural transformations of functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$.

The functor $\mathcal{Y}$ extends to a 2-functor: given objects $(f\colon \mathcal{D} \to \mathcal{C})$ and $(g\colon \mathcal{D}' \to \mathcal{C})$ and morphisms $(q,\theta), (q',\theta')\colon f \to g$ in $\mathrm{T}(\mathcal{C})$, then a 2-morphism

$$\alpha\colon (q,\theta) \Rightarrow (q',\theta')$$

is sent by $\mathcal{Y}$ to the modification whose component at an object x of $\mathcal{C}$ is the natural transformation

$$\mathcal{Y}(q,\theta)_x \Rightarrow \mathcal{Y}(q',\theta')_x$$

that is defined by the components of α. The 2-functor $\mathcal{Y}$ is again full and faithful (*categorified Cat-version of the Yoneda embedding*).

For every functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$, there is an equivalence of categories

$$\Lambda_c(F)\colon \underline{\mathrm{Hom}}(\mathcal{Y}_e(c), F) \xrightarrow{\cong} F(c)$$

(*categorified Yoneda lemma for Cat-presheaves*), which is defined on objects by $(\theta\colon \mathcal{Y}_e(c) \to F) \mapsto \theta_c(\mathrm{id}_c) \in \mathrm{Ob}F(c)$.

The Grothendieck construction defines a (2-)functor

$$\textstyle\int\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}) \to \mathrm{Cat}_{/\mathcal{C}}, \quad (F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}) \mapsto (p_F\colon \int F \to \mathcal{C}).$$

The objects of $\int F$ are pairs of objects $(c \in \mathrm{Ob}\mathcal{C}; z \in \mathrm{Ob}F(c))$. A morphism from $(c; z)$ to $(c'; z')$ in $\int F$ is a pair of morphisms $u\colon c \to c'$ in $\mathcal{C}$ and $h\colon z \to F(u)(z')$ in $F(c)$.

Let $\mathrm{Fib}^{\mathrm{cart}}(\mathcal{C})$ be the 2-subcategory of $\mathrm{Cat}_{/\mathcal{C}}$ whose objects are the *cartesian fibrations over* $\mathcal{C}$, the 1-morphisms are those functors over $\mathcal{C}$ which preserve cartesian morphisms, and the 2-morphisms are as in $\mathrm{Cat}_{/\mathcal{C}}$. The functor $\int$ factors through the inclusion of the subcategory $\mathrm{Fib}^{\mathrm{cart}}(\mathcal{C}) \hookrightarrow \mathrm{Cat}_{/\mathcal{C}}$ and restricts to an equivalence of 2-categories $\boxed{\int\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}) \simeq \mathrm{Fib}^{\mathrm{cart}}(\mathcal{C})}$. Then the composite (2-)functor

$$\mathrm{T}(\mathcal{C}) \xrightarrow{\mathcal{Y}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}) \xrightarrow{\int} \mathrm{Fib}^{\mathrm{cart}}(\mathcal{C})$$

is also full and faithful. For example, there are isomorphisms of categories

$$(g_{c/} \cong)\ \mathrm{T}(\mathcal{C})(e \xrightarrow{c} \mathcal{C}, g) \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathrm{Cat}^{\mathcal{C}^{\mathrm{op}}}}(\mathcal{Y}_e(c), \mathcal{Y}(g)) \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathrm{Fib}^{\mathrm{cart}}(\mathcal{C})}(\textstyle\int \mathcal{Y}_e(c), \int \mathcal{Y}(g)).$$

Let Cat_∞ denote (a model for) the $(\infty,1)$-category of ∞-categories and let $\mathrm{Cat}_{\infty/\mathcal{C}}$ be the ∞-category of functors over $\mathcal{C}$.

Let $\mathrm{Fib}^{\mathrm{cart}}(\mathcal{C}) \subset \mathrm{Cat}_{\infty/\mathcal{C}}$ be the ∞-subcategory of cartesian fibrations over $\mathcal{C}$. The morphisms in $\mathrm{Fib}^{\mathrm{cart}}(\mathcal{C})$ are given by functors over $\mathcal{C}$ that preserve the cartesian edges.

Given objects $(f\colon \mathcal{D} \to \mathcal{C})$ and $(g\colon \mathcal{D}' \to \mathcal{C})$ in $\mathrm{T}(\mathcal{C})$, there is a natural bijection

$$\mathrm{Hom}_{\mathrm{T}(\mathcal{C})}(f,g) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Cat}_{/\mathcal{C}}}(f, \textstyle\int \mathcal{Y}(g))$$

which sends (q,θ) to the functor $\mathcal{D} \to \int \mathcal{Y}(g)$ over $\mathcal{C}$ such that

$$d \mapsto (f(d); q(d), f(d) \xrightarrow{\theta_d} g(q(d))).$$

The inverse is given by composition with the following morphism in $\mathrm{T}(\mathcal{C})$,

$$(r\colon \textstyle\int \mathcal{Y}(g) \to \mathcal{D}', \theta\colon p_{\mathcal{Y}(g)} \Rightarrow g \circ r)$$

where $r(x; d', x \to g(d')) = d'$.

Given a functor $(g\colon \mathcal{D}' \to \mathcal{C})$ between ∞-categories, there is an *associated free cartesian fibration* over $\mathcal{C}$

$$\widetilde{g}\colon \mathcal{E}(g) \colon= \mathcal{D}' \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} \to \mathcal{C}$$
$$(d', x \to g(d')) \mapsto x.$$

The ∞-category of morphisms from $(f\colon \mathcal{D} \to \mathcal{C})$ to $(\widetilde{g}\colon \mathcal{E}(g) \to \mathcal{C})$ is given by the fiber of the associated functor

$$\begin{array}{c} \mathrm{Fun}(\mathcal{D}, \mathcal{D}') \times_{\mathrm{Fun}(\mathcal{D},\mathcal{C})} \mathrm{Fun}(\mathcal{D}, \mathcal{C}^{\Delta^1}) \\ \downarrow \\ \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \end{array}$$

over the object $f\colon \mathcal{D} \to \mathcal{C}$.

The cartesian fibration

$$p_{\mathcal{Y}(f)}\colon \textstyle\int \mathcal{Y}(f) \to \mathcal{C}$$

is *free*: there is a canonical functor $\iota\colon \mathcal{D} \to \int \mathcal{Y}(f)$ over $\mathcal{C}$, defined on objects by

$$d \mapsto (f(d); d, \mathrm{id}_{f(d)}),$$

such that the functor given by composition with ι

$$\underline{\mathrm{Hom}}_{\mathrm{Fib}^{\mathrm{cart}}(\mathcal{C})}(\textstyle\int \mathcal{Y}(f), (\mathcal{E} \xrightarrow{p} \mathcal{C})) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathrm{Cat}_{/\mathcal{C}}}(f, (\mathcal{E} \xrightarrow{p} \mathcal{C}))$$

is an equivalence of categories for every cartesian fibration p (e.g., $p = p_{\int \mathcal{Y}(g)}$).

$\mathrm{Cat}_{\infty/\mathcal{C}}$ can also be regarded as an $(\infty,2)$-category, using that Cat_∞ is also an $(\infty,2)$-category.

The inclusion of $(\infty,1)$-categories

$$\mathrm{Fib}^{\mathrm{cart}}(\mathcal{C}) \hookrightarrow \mathrm{Cat}_{\infty/\mathcal{C}}$$

admits a left adjoint

$$\mathcal{E}\colon \mathrm{Cat}_{\infty/\mathcal{C}} \to \mathrm{Fib}^{\mathrm{cart}}(\mathcal{C}).$$

There is an *unstraightening functor* $\int\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty) \to \mathrm{Cat}_{\infty/\mathcal{C}}$ which restricts to an equivalence $\boxed{\int\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty) \simeq \mathrm{Fib}^{\mathrm{cart}}(\mathcal{C})}$ and an (inverse) *straightening functor*

$$\langle \mathcal{Y} \rangle\colon \mathrm{Fib}^{\mathrm{cart}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)$$

which is an equivalence of $(\infty,1)$-categories (*Cat_∞-version of the Yoneda embedding*). The restriction of $\langle \mathcal{Y} \rangle$ along $\mathcal{Y}_e\colon \mathcal{C} \to \mathrm{Fib}^{\mathrm{cart}}(\mathcal{C})$, $c \mapsto (\mathcal{C}_{/c} \to \mathcal{C})$, is identified with the Yoneda embedding $\mathrm{y}\colon \mathcal{C} \to \mathcal{P}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_\infty)$.