ACCESSIBLE ∞-CATEGORIES

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1. Preliminaries

1.1. Locally small ∞-categories.

Definition 1.1.1. An ∞-category $\mathcal{C}$ is called *locally small* if for any small subset $S \subset \mathcal{C}_0$ of objects in $\mathcal{C}$, the full subcategory generated by S is essentially small.

Proposition 1.1.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories such that $\mathcal{C}$ is essentially small and $\mathcal{D}$ is locally small. Then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is locally small as well. In particular, the ∞-category of presheaves $\mathcal{P}(\mathcal{C})$ on an essentially small ∞-category $\mathcal{C}$ is locally small.*

Proof. See [2, Example 5.4.1.8]. □

Remark 1.1.3. The ∞-category $\mathrm{Ind}_\kappa(\mathcal{C})$ (as a full subcategory of $\mathcal{P}(\mathcal{C})$) is locally small if $\mathcal{C}$ is essentially small.

1.2. Retracts and Idempotents.

Definition 1.2.1. Let Idem denote the nerve of the monoid $\{\mathrm{id}, e\}$ with multiplication law $e \cdot e = e$. Then we refer to $\mathrm{Fun}(\mathsf{Idem}, \mathcal{C})$ as the *∞-category of idempotents* in $\mathcal{C}$. A discussion about idempotents can be found in [2, 4.4.5]. This ∞-category is essentially small if $\mathcal{C}$ is essentially small.

Remark 1.2.2. Note that for an essentially small ∞-category $\mathcal{C}$, any retract of a representable presheaf (in $\mathrm{Ind}_\kappa(\mathcal{C})$) gives rise to an idempotent $p\colon \mathsf{Idem} \to \mathcal{C}$. We can recover the retract by taking the colimit of $p\colon \mathsf{Idem} \to \mathcal{C} \subseteq \mathrm{Ind}_\kappa(\mathcal{C})$ (see [2, Corollary 4.4.5.14] and [2, Remark 5.3.1.9]). Any such colimit is κ-compact in $\mathrm{Ind}_\kappa(\mathcal{C})$ because representable presheaves in $\mathrm{Ind}_\kappa(\mathcal{C})$ are κ-compact and κ-compact objects are closed under retracts. Thus, we have a functor $\mathrm{Fun}(\mathsf{Idem}, \mathcal{C}) \to \mathrm{Ind}_\kappa(\mathcal{C})^\kappa$ defined by the colimit.

2. Accessible ∞-categories

2.1. Basic definitions and properties.

Definition 2.1.1. Let κ be a regular cardinal. An ∞-category $\mathcal{C}$ is called *κ-accessible* if there exists an essentially small ∞-category $\mathcal{C}'$ and an equivalence of ∞-categories

$$\mathrm{Ind}_\kappa(\mathcal{C}') \xrightarrow{\simeq} \mathcal{C}.$$

$\mathcal{C}$ is called *accessible* if there exists a regular cardinal κ such that $\mathcal{C}$ is κ-accessible.

The following proposition proves some equivalent characterizations of accessible ∞-categories.

Proposition 2.1.2. *Let $\mathcal{C}$ be an ∞-category and κ a regular cardinal. Then the following are equivalent.*

(1) $\mathcal{C}$ is κ-accessible.
(2) $\mathcal{C}$ admits κ-filtered colimits and the full subcategory $\mathcal{C}^\kappa$ of κ-compact objects is essentially small and generates $\mathcal{C}$ under κ-filtered colimits.
(3) $\mathcal{C}$ admits κ-filtered colimits and there exists a full and essentially small subcategory $\mathcal{C}'$ of κ-compact objects, which generates $\mathcal{C}$ under κ-filtered colimits.

PROOF. See [2, Proposition 5.4.2.2].
(1) $\Rightarrow$ (2) : $\mathcal{C}$ admits κ-filtered colimits by [2, Proposition 5.3.5.3]. We know that there is a small ∞-category $\mathcal{C}'$ such that $\mathrm{Ind}_\kappa(\mathcal{C}') \simeq \mathcal{C}$. We claim that $\mathrm{Ind}_\kappa(\mathcal{C}')^\kappa$ is essentially small: any $f \in \mathrm{Ind}_\kappa(\mathcal{C}')^\kappa$ classifies a right fibration $p\colon \widetilde{\mathcal{C}'}(f) \to \mathcal{C}'$ where $\widetilde{\mathcal{C}'}(f)$ is κ-filtered, and f can be identified with the colimit of the canonical diagram

$$\widetilde{\mathcal{C}'}(f) \xrightarrow{p} \mathcal{C}' \xrightarrow{j} \mathrm{Ind}_\kappa(\mathcal{C}').$$

Therefore, choosing a representative of the 0-simplex:

$$\Delta^0 \xrightarrow{\mathrm{id}_f} \mathrm{map}(f,f) \simeq \mathrm{map}(f, \mathrm{colim}\, j \circ p) \simeq \mathrm{colim}\, \mathrm{map}(f, j \circ p)$$

shows that f is a retract of some representable presheaf $j(c)$, $c \in \mathcal{C}'$. Thus the claim follows from 1.2.2 and 1.2.1.

It remains to show that $\mathrm{Ind}_\kappa(\mathcal{C}')^\kappa$ generates $\mathrm{Ind}_\kappa(\mathcal{C}')$ under κ-filtered colimits. But the Yoneda embedding factors as:

$$\mathcal{C}' \xrightarrow{j} \mathrm{Ind}_\kappa(\mathcal{C}')^\kappa \longrightarrow \mathrm{Ind}_\kappa(\mathcal{C}')$$

So the claim follows since $\mathcal{C}'$ already generates $\mathrm{Ind}_\kappa(\mathcal{C}')$ under κ-filtered colimits.
(2) $\Rightarrow$ (3): obvious.
(3) $\Rightarrow$ (1) : The assumptions give exactly a factorization of the form

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[dr, "j"'] & \mathcal{C}^\kappa \arrow[r] & \mathcal{C} \\
& \mathrm{Ind}_\kappa(\mathcal{C}') \arrow[ur, dashed] &
\end{tikzcd}$$

such that the upper composition is fully faithful and $\mathcal{C}'$ generates $\mathcal{C}$ under κ-filtered colimits. Then we can apply [2, Proposition 5.3.5.11.(2)] to conclude that the canonical extension of this functor along j is an equivalence. □

Given a regular cardinal κ, we define a category Acc_κ enriched in Kan complexes as follows. The objects of Acc_κ are given by κ-accessible ∞-categories. The mapping space from $\mathcal{C}$ to $\mathcal{D}$ in Acc_κ is the maximal ∞-groupoid of the full subcategory $\mathrm{Fun}_\kappa(\mathcal{C},\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C},\mathcal{D})$ that is spanned by those functors which preserve κ-filtered colimits and κ-compact objects. We refer to the associated ∞-category $\mathrm{N}_\Delta(\mathsf{Acc}_\kappa)$ as the ∞-category of κ-accessible ∞-categories. We also denote by Cat_∞ the ∞-category of essentially small ∞-categories.

Proposition 2.1.3. *The functor*

$$\mathrm{N}_\Delta(\mathsf{Acc}_\kappa) \longrightarrow \mathsf{Cat}_\infty$$

induced by the functor (between enriched categories) which assigns to a κ-accessible ∞-category $\mathcal{C}$ the ∞-category of κ-compact objects $\mathcal{C}^\kappa$, is fully faithful.

Proof. See [2, Proposition 5.4.2.15].
There is an equivalence

$$\mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(\mathcal{C}^\kappa),\mathcal{D})\simeq \mathrm{Fun}(\mathcal{C}^\kappa,\mathcal{D}^\kappa)$$

as a consequence of the universal property of $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$ [2, Proposition 5.3.5.10]. This restricts to a homotopy equivalence of the associated ∞-groupoids. □

Remark 2.1.4. The essential image of the fully faithful functor in Proposition 2.1.3 is the ∞-category of essentially small idempotent complete ∞-categories. (See [2, Proposition 5.4.2.15].) Moreover, the functor $\mathrm{Ind}_\kappa(-)\colon \mathsf{Cat}_\infty \to \mathrm{N}_\Delta(\mathsf{Acc}_\kappa)$ exhibits $\mathrm{N}_\Delta(\mathsf{Acc}_\kappa)$ as a localization of Cat_∞ [2, Proposition 5.4.2.17]. As a consequence, $\mathrm{N}_\Delta(\mathsf{Acc}_\kappa)$ inherits small limits from Cat_∞. However, these do not agree with the limits of the underlying ∞-categories in general. We will return to the question of the existence of pullbacks of accessible ∞-categories in Section 3.

2.2. **Raising the rank of accessibility.**

Notation 2.2.1. Let κ and τ be regular cardinals. We write $\kappa \prec\!\prec \tau$ if for any cardinals $\kappa_0 \prec \kappa$ and $\tau_0 \prec \tau$, we have $\tau_0^{\kappa_0} \prec \tau$.

Lemma 2.2.2. *Let κ and τ be regular cardinals such that $\kappa \prec\!\prec \tau$. Then any κ-filtered partially ordered set $\mathcal{J}$ can be written as a union of subsets which are κ-filtered and τ-small. Moreover the partially ordered set given by those subsets is τ-filtered.*

Proof. See [2, Lemma 5.4.2.8]. □

Construction 2.2.3. Let $f\colon \mathrm{N}(\mathcal{J}) \to \mathcal{C}$ be a diagram where $\mathcal{J}$ is a κ-directed partially ordered set. Using Lemma 2.2.2, given any cardinal $\kappa \prec\!\prec \tau$, there is a τ-filtered partially ordered set $\mathcal{I}$ which consists of subsets of $\mathcal{J}$ which are κ-filtered and τ-small. This decomposition of $\mathcal{J}$ can be used to define a functor $F\colon \mathcal{I} \to \mathsf{sSet}_{/\,\mathrm{N}(\mathcal{J})}$, which sends a subset $(S \subseteq \mathcal{J})$ in $\mathcal{I}$ to the map between the nerves induced by the inclusion. Assuming that $\mathcal{C}$ admits κ-filtered colimits, we may find for any subset $(S \subseteq \mathcal{J})$ of $\mathcal{I}$ a colimit cone of the respective diagram:

$$\mathrm{N}(S) \to \mathrm{N}(\mathcal{J}) \to \mathcal{C}.$$

Furthermore, if the original diagram f consists of τ-compact objects, these colimits will remain τ-compact (because τ-compact objects are closed under τ-small colimits – see the proof of Proposition 2.2.4 below). Using the method explained in [2, Proposition 4.2.3.4] combined with [2, Proposition 4.2.3.8], we can assemble these colimit cones to a diagram $\mathrm{N}(\mathcal{I}) \to \mathcal{C}$ which has the same colimit as f. In other words, if $\kappa \prec\!\prec \tau$, then we can rewrite any κ-directed colimit of κ-compact (or τ-compact) objects as a τ-directed colimit of τ-compact objects which are obtained as κ-filtered τ-small colimits of κ-compact (or τ-compact) objects.

Proposition 2.2.4. *Let $\mathcal{C}$ be a κ-accessible ∞-category. Then $\mathcal{C}$ is τ-accessible for any $\kappa \prec\!\prec \tau$.*

Proof. See [2, Proposition 5.4.2.9].
We will use Proposition 2.1.2(3). Our candidate for an essentially small generating subcategory $\mathcal{C}'$ will be the full subcategory spanned by colimits of diagrams in $\mathcal{C}^\kappa$ which are τ-small and κ-filtered.

First we observe that the collection of τ-small simplicial sets is indeed a set. So the collection of equivalence classes of τ-small diagrams in $\mathcal{C}^{\kappa}$ is a set as well. Since $\mathcal{C}'$ is locally small, we conclude that $\mathcal{C}'$ is indeed essentially small.

Then we claim that $\mathcal{C}'$ consists of τ-compact objects. By definition, an object $c' \in \mathcal{C}'$ can be written as $c' \simeq \operatorname{colim}_K p$, where $p\colon K \to \mathcal{C}^{\kappa}$ is a τ-small diagram. Then, for any τ-filtered diagram $q\colon S \to \mathcal{C}$, using that τ-small limits commute with τ-filtered colimits and that any κ-compact object is also τ-compact, we obtain equivalences:

$$\operatorname{colim}_S \operatorname{map}(c', q(-)) \simeq \operatorname{colim}_S \lim_K \operatorname{map}(p(-), q(-)) \simeq \\ \lim_K \operatorname{colim}_S \operatorname{map}(p(-), q(-)) \simeq \lim_K \operatorname{map}(p(-), \operatorname{colim}_S q) \simeq \\ \simeq \operatorname{map}(c', \operatorname{colim}_S q).$$

It remains to show that $\mathcal{C}'$ generates $\mathcal{C}$ under τ-filtered colimits. This is a consequence of Construction 2.2.3. □

2.3. **Uniformization.**

The following proposition generalizes [1, 2.19] to accessible ∞-categories.

Proposition 2.3.1. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a κ-accessible functor between κ-accessible ∞-categories. Then there exists a cardinal $\kappa \prec\!\prec \tau$ such that p sends τ-compact objects in $\mathcal{C}$ to τ-compact objects in $\mathcal{D}$.*

Proof. Since $\mathcal{D} \simeq \operatorname{Ind}_{\kappa}(\mathcal{D}')$ for some small ∞-category $\mathcal{D}'$, any object $d \in \mathcal{D}$ can be expressed as the colimit of a (small) κ-filtered diagram in $\mathcal{D}$ with values in $\mathcal{D}'$. In particular, $d \in \mathcal{D}$ is τ-compact for some τ (for example, we may choose a regular cardinal $\tau \succ \kappa$ for which the given κ-filtered ∞-category is τ-small). The full subcategory of κ-compact objects $\mathcal{C}^{\kappa} \subseteq \mathcal{C}$ is (essentially) small, therefore we may find $\tau \succ\!\succ \kappa$ such that $p(c)$ is τ-compact for every κ-compact object $c \in \mathcal{C}$ (for example, we may choose $\tau \succ\!\succ \kappa$ to be greater than the cardinalities of the κ-filtered ∞-categories, which are associated, as indicated above, to the objects in the image of $\mathcal{C}^{\kappa}$ under p). Then we conclude that p sends κ-compact objects to τ-compact objects.

Construction 2.2.3 together with the last part of Proposition 2.1.2(1) ⇒ (2) show that any object $c \in \mathcal{C}^{\tau}$ is a retract of a κ-filtered and τ-small colimit in $\mathcal{C}^{\kappa}$. Since p commutes with κ-filtered colimits, and τ-small colimits of τ-compact objects are again τ-compact, the image of this colimit under p is also τ-compact in $\mathcal{D}$. Since τ-compact objects are stable under retracts, it follows that τ has the required properties. □

3. Pullbacks of Accessible ∞-categories

The main goal of this section is to prove that the ∞-category of accessible ∞-categories admits pullbacks and that these agree with the pullbacks of underlying ∞-categories (= homotopy pullbacks of ∞-categories in the sense of the Joyal model structure).

3.1. **Colimits in pullbacks of ∞-categories.**

Consider a homotopy pullback of ∞-categories in sSet equipped with the Joyal model structure:

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "q'"] \arrow[d, "p'"'] & \mathcal{C} \arrow[d, "p"] \\ \mathcal{D}' \arrow[r, "q"'] & \mathcal{D}. \end{tikzcd}$$

Note that this situation arises when the diagram is a strict pullback and p or q is an isofibration (= categorical fibration). Moreover, every homotopy pullback may be replaced (up to Joyal equivalence) by a strict pullback of this form and, for simplicity, we will often assume that the homotopy pullback is also a strict pullback of simplicial sets.

Proposition 3.1.1. *Suppose that $\mathcal{C}$ and $\mathcal{D}'$ admit initial objects and these are preserved by p and q. Then an object $c' \in \mathcal{C}'$ is initial if and only if $p'(c') \in \mathcal{D}'$ and $q'(c') \in \mathcal{C}$ are initial. Furthermore, there exists an initial object $c' \in \mathcal{C}'$.*

Proof. See [2, Lemma 5.4.5.2].
We will first prove that there exists an object $c' \in \mathcal{C}$, which maps to initial objects in $\mathcal{C}$ and $\mathcal{D}'$. Let $d' \in \mathcal{D}'$ and $c \in \mathcal{C}$ be initial objects. Since $q(d')$ and $p(c)$ are also initial, they must be equivalent. Assuming that q is an isofibration (as we may do without loss of generality), the equivalence $q(d') \simeq p(c)$ can be lifted to an equivalence $d' \simeq \widetilde{d'}$ in $\mathcal{D}'$. Then the pair $(\widetilde{d'}, c)$ defines an object in $c' \in \mathcal{C}'$ which maps to the initial objects $\widetilde{d'}$ and c respectively.

For any other object $z \in \mathcal{C}'$, we have a homotopy pullback of mapping spaces (in the Kan-Quillen model structure):

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}'}(c', z) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{C}}(q'(c'), q'(z)) \arrow[d] \\ \mathrm{map}_{\mathcal{D}'}(p'(c'), p'(z)) \arrow[r] & \mathrm{map}_{\mathcal{D}}(qp'(c'), qp'(z)). \end{tikzcd}$$

It follows that $\mathrm{map}_{\mathcal{C}'}(c', z)$ is contractible, since every other mapping space in the square is contractible. This shows that $c' \in \mathcal{C}'$ is initial, thus, proving the second assertion and the "if"-direction of the first assertion. Since any two initial objects are equivalent, it follows that p' and q' also preserve initial objects, which then concludes the proof of the first assertion. □

Corollary 3.1.2. *Let K be a simplicial set and suppose that $\mathcal{C}$ and $\mathcal{D}$ admit all K–indexed colimits and p and q preserve these. Then $\widetilde{f}\colon K^{\triangleright} \to \mathcal{C}'$ is a colimit cone of its restriction $f\colon K \to \mathcal{C}'$ if and only if $p' \circ \widetilde{f}$ and $q' \circ \widetilde{f}$ are colimit cones of $p' \circ f$ and $q' \circ f$. Furthermore any diagram $f\colon K \to \mathcal{C}'$ admits a colimit cone.*

Proof. See [2, Lemma 5.4.5.5].
Using Proposition 3.1.1 and the fact that a diagram $f\colon K \to \mathcal{C}$ admits a colimit if and only if $\mathcal{C}_{f/}$ has an initial object, it suffices to check that the induced pullback square

$$\begin{tikzcd} \mathcal{C}'_{f/} \arrow[r] \arrow[d] & \mathcal{C}_{q'f/} \arrow[d] \\ \mathcal{D}'_{p'f/} \arrow[r] & \mathcal{D}_{qp'f/} \end{tikzcd}$$

is a homotopy pullback of ∞-categories. For this, it suffices to check that the functor $\mathcal{D}'_{p'f/} \to \mathcal{D}_{qp'f/}$ is an isofibration (=categorical fibration), that is, we need to solve the following equivalent lifting problems:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "\simeq"'] & \mathcal{D}'_{p'f/} \arrow[d] & & K \star A \arrow[r] \arrow[d, hook, "\simeq"'] & \mathcal{D}' \arrow[d, two heads, "q"] \\
B \arrow[r] & \mathcal{D}_{qp'f/} & & K \star B \arrow[r] \arrow[ur, dashed] & \mathcal{D}.
\end{tikzcd}$$

The square on the right admits a lift because $K \star -$ preserves monomorphisms and Joyal equivalences, and q is an isofibration (by assumption). □

The following proposition gives a criterion for the identification of κ-compact objects in pullbacks of ∞-categories.

Proposition 3.1.3. *Let κ be a regular cardinal such that $\mathcal{C}$ and $\mathcal{D}'$ admit κ-filtered colimits and p and q preserve these. If for some object $c' \in \mathcal{C}'$, the objects $q'(c')$, $p'(c')$ and $pq'(c') = qp'(c')$ are all κ-compact, then c' is also κ-compact.*

Proof. See [2, Lemma 5.4.5.7].
Let $f\colon K \to \mathcal{C}'$ be a κ-filtered diagram. We know from Corollary 3.1.2 that f admits a colimit and the functors p' and q' preserve this colimit. Since $q'(c'), p'(c')$ and $qp'(c')$ are κ-compact, we have equivalences as follows,

$$\mathrm{map}_{\mathcal{C}}(q'(c'), q'(\mathrm{colim}_K f)) \simeq \mathrm{map}_{\mathcal{C}}(q'(c'), \mathrm{colim}_K q'f) \simeq \mathrm{colim}_K \mathrm{map}_{\mathcal{C}}(q'(c'), q'f),$$

and similarly for the other two objects. Hence it suffices to check that we have a homotopy pullback square of spaces (in the Kan-Quillen model structure):

$$\begin{tikzcd}
\mathrm{colim}_K \mathrm{map}_{\mathcal{C}'}(c', f) \arrow[r] \arrow[d] & \mathrm{colim}_K \mathrm{map}_{\mathcal{C}}(q'(c'), q'f) \arrow[d] \\
\mathrm{colim}_K \mathrm{map}_{\mathcal{D}'}(p'(c'), p'f) \arrow[r] & \mathrm{colim}_K \mathrm{map}_{\mathcal{D}}(qp'(c'), qp'f).
\end{tikzcd}$$

This follows from the fact that pullbacks in the ∞-category of spaces commute with filtered colimits. □

3.2. **Pullbacks of accessible ∞-categories.** Our next goal is to prove the following theorem:

Theorem 3.2.1. *Let*

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "q'"] \arrow[d, "p'"'] & \mathcal{C} \arrow[d, "p"] \\
\mathcal{D}' \arrow[r, "q"'] & \mathcal{D}
\end{tikzcd}$$

be a homotopy pullback of ∞-categories (with respect to the Joyal model structure) where the ∞-categories $\mathcal{C}, \mathcal{D}$ and $\mathcal{D}'$ are accessible and the functors p and q are accessible.

Then $\mathcal{C}'$ is also accessible. Furthermore, a functor $f\colon \mathcal{E} \to \mathcal{C}'$ is accessible if and only if the functors $p' \circ f$ and $q' \circ f$ are accessible.

By Proposition 2.2.4, we may assume that the ∞-categories $\mathcal{C}, \mathcal{D}, \mathcal{D}'$ and the functors p and q are κ-accessible for some regular cardinal κ. Then Corollary 3.1.2 shows that $\mathcal{C}'$ admits κ-filtered colimits and also proves the second claim in the theorem. By Proposition 2.3.1, there is $\tau \succ\succ \kappa$ such that in addition p and q send τ-compact objects to τ-compact objects.

Therefore it suffices to prove the following more refined statement which also emphasizes the role of the two cardinals κ and τ (cf. [3, 2.2]).

Theorem 3.2.2. *Let $\kappa \prec \tau$ be regular cardinals and let*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "q'"] \arrow[d, "p'"'] & \mathcal{C} \arrow[d, "p"] \\ \mathcal{D}' \arrow[r, "q"'] & \mathcal{D} \end{tikzcd}$$

be a homotopy pullback of ∞-categories (with respect to the Joyal model structure). Suppose that:

- *(i) $\mathcal{C}, \mathcal{D}$ and $\mathcal{D}'$ are τ-accessible and admit κ-filtered colimits.*
- *(ii) p and q preserve κ-filtered colimits and τ-compact objects.*

Then $\mathcal{C}'$ is τ-accessible and admits κ-filtered colimits.

Outline of the Proof. Consider the homotopy pullback of essentially small ∞-categories:

$$\begin{tikzcd} \mathcal{C}'' \arrow[r, "q'"] \arrow[d, "p'"'] & \mathcal{C}^\tau \arrow[d, "p"] \\ \mathcal{D}'^\tau \arrow[r, "q"'] & \mathcal{D}^\tau. \end{tikzcd}$$

We may view $\mathcal{C}''$ as an essentially small full subcategory of $\mathcal{C}'$. As a consequence of Proposition 3.1.3, the pullback $\mathcal{C}''$ consists of τ-compact objects in $\mathcal{C}'$. The ∞-category $\mathcal{C}''$ is our candidate of an essentially small full subcategory to generate $\mathcal{C}'$ under τ-filtered colimits. Proving this would then allow us to conclude that $\mathcal{C}'$ is τ-accessible using the characterization of Theorem 2.1.2(3).

Let $c' \in \mathcal{C}'$ be an object and $d' = p'(c'), c = q'(c'), d = pq'(c')$. Then we obtain a homotopy pullback of ∞-categories:

$$\begin{tikzcd} \mathcal{C}''_{/c'} \arrow[r, "q'"] \arrow[d, "p'"'] & \mathcal{C}^\tau_{/c} \arrow[d, "p"] \\ \mathcal{D}'^\tau_{/d'} \arrow[r, "q"'] & \mathcal{D}^\tau_{/d}. \end{tikzcd}$$

The three outer ∞-categories are τ-filtered as a consequence of the assumption that $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{D}'$ are τ-accessible[1].

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[1] To see that $\mathcal{C}^\tau_{/c}$ is τ-filtered, consider a τ-small diagram $f\colon K \to \mathcal{C}^\tau_{/c}$ and write c as a colimit of a τ-filtered diagram $\mathcal{J} \to \mathcal{C}^\tau$, $j \mapsto c_j$, of τ-compact objects. Then it suffices to prove that f, regarded as a cone in $\mathcal{C}$, factors through some stage $(c_j \to c)$ of this τ-filtered diagram. But note that f is canonically identified with a point in the space $\lim_{i \in K^{\mathrm{op}}} \mathrm{map}_{\mathcal{C}}(f(i), c)$, and then the equivalences

$$\lim_{K^{\mathrm{op}}} \mathrm{map}(f(i), c) \simeq \lim_{K^{\mathrm{op}}} \mathrm{colim}_{\mathcal{J}} \mathrm{map}(f(i), c_j) \simeq \mathrm{colim}_{\mathcal{J}} \lim_{K^{\mathrm{op}}} \mathrm{map}(f(i), c_j)$$

show the required factorization. The other two cases are treated in the same way.

In addition, the three outer ∞-categories in the pullback define canonical τ-filtered diagrams of τ-compact objects in the respective ∞-categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{D}'$, with colimits the objects c, d, and d', respectively. (Explicitly, in the case of $\mathcal{C}^{\tau}_{/c}$ for example, this says that the canonical cone with cone object c

$$(\mathcal{C}^{\tau}_{/c})^{\triangleright} \to \mathcal{C}_{/c} \to \mathcal{C}$$

defines a colimit diagram. To see this, recall that the inclusion $\mathcal{C}^{\tau} \hookrightarrow \mathcal{C}$ extends to an equivalence $\mathrm{Ind}_{\tau}(\mathcal{C}^{\tau}) \simeq \mathcal{C}$ – see the proof of Proposition 2.1.2 – and then note that the canonical functor $\mathcal{C}^{\tau}_{/c} \to \mathcal{C}^{\tau}$ is the right fibration which corresponds to the presheaf $(\mathcal{C}^{\tau})^{\mathrm{op}} \to \mathcal{S},\ x \mapsto \mathrm{map}_{\mathcal{C}}(x, c)$, i.e., the image of c under the restricted Yoneda functor $\mathcal{C} \to \mathcal{P}(\mathcal{C}^{\tau})$.)

Then it suffices to prove the following two claims:

Claim (A). The pullback $\mathcal{C}''_{/c'}$ is τ-filtered.

Claim (B). The functors $p' \colon \mathcal{C}''_{/c'} \to \mathcal{D}'^{\tau}_{/d'}$ and $q' \colon \mathcal{C}''_{/c'} \to \mathcal{C}^{\tau}_{/c}$ are cofinal.

Indeed, assuming (A) and (B), we have a τ-filtered diagram

$$\mathcal{C}''_{/c'} \to \mathcal{C}'' \to \mathcal{C}'$$

whose colimit is given by c', as a consequence of Corollary 3.1.2 and (B). As explained above, this suffices to conclude the first claim of Theorem 3.2.2. The second claim of Theorem 3.2.2 is immediate from Corollary 3.1.2. This completes our outline of the proof. ☐

The proofs of Claims (A) and (B) require intricate cofinality arguments and will be given in the Appendix below.

Appendix A. The proofs of Claims (A) and (B)

We will make use of the following proposition.

Proposition A.1. *Let τ be a regular cardinal. Let $\mathcal{C}$ be a τ-accessible ∞-category and $f \colon K \to \mathcal{C}^{\tau}$ a τ-small diagram in $\mathcal{C}$. Then $\mathcal{C}_{f/}$ is τ-accessible and the τ-compact objects in $\mathcal{C}_{f/}$ are exactly those which are sent to τ-compact objects under the canonical functor $\mathcal{C}_{f/} \to \mathcal{C}$.*

Proof. The ∞-category $\mathcal{C}_{f/}$ admits τ-filtered colimits [2, Lemma 5.4.5.14]. Consider the full subcategory $\mathcal{C}^{\tau}_{f/} = \mathcal{C}_{f/} \times_{\mathcal{C}} \mathcal{C}^{\tau}$ of $\mathcal{C}_{f/}$. Note that this is essentially small. Every object $(K^{\triangleright} \to \mathcal{C}^{\tau})$ in $\mathcal{C}^{\tau}_{f/}$ is τ-compact in $\mathcal{C}_{f/}$ [2, Lemma 5.4.5.13]. To see this, first recall that $\mathcal{C}_{f/}$ is equivalent to the (homotopy) pullback

$$\begin{tikzcd}
\mathcal{C}^{f/} \arrow[r] \arrow[d] & \mathrm{Fun}(K \times \Delta^1 / K \times \{1\}, \mathcal{C}) \arrow[d] \\
\Delta^0 \arrow[r, "f"] & \mathrm{Fun}(K \times \{0\}, \mathcal{C}).
\end{tikzcd}$$

Then, using Proposition 3.1.3, we may reduce the claim to the statement that a pointwise τ-compact object in $\mathrm{Fun}(L, \mathcal{C})$ is τ-compact, assuming that L is τ-small. This statement can be shown by induction on the skeletal filtration of L using Proposition 3.1.3.

Therefore, by Proposition 2.1.2, it suffices to show that $\mathcal{C}^{\tau}_{f/}$ generates $\mathcal{C}_{f/}$ under τ-filtered colimits. Let $C = (K^{\triangleright} \to \mathcal{C})$ be an object of $\mathcal{C}_{f/}$ and let $c \in \mathcal{C}$ denote the image of the cone point of $K^{\triangleright}$. There is a τ-directed poset $\mathcal{J}$ and a diagram

$$\phi\colon \mathcal{J} \to \mathcal{C}^{\tau},\ j \overset{\phi}{\mapsto} c_j,$$

such that $c \simeq \operatorname{colim}_{\mathcal{J}} \phi$ – this uses [2, Proposition 5.3.1.18]. Note that the object $C \in \mathcal{C}_{f/}$ corresponds canonically to a point in

$$\operatorname{map}_{\mathcal{C}^K}(f, \underline{c}) \simeq \lim_{K^{\mathrm{op}}} \operatorname{map}_{\mathcal{C}}(f(-), c)$$

where $\underline{c}\colon K \to \mathcal{C}$ denotes the constant functor at $c \in \mathcal{C}$. Using that f takes values in τ-compact objects in $\mathcal{C}$, K is τ-small, and the fact that τ-filtered colimits commute with τ-small limits in the ∞-category of spaces, it follows that the cone $C\colon K^{\triangleright} \to \mathcal{C}$ factors through some cone $C_i\colon K^{\triangleright} \to \mathcal{C}^{\tau}$ with cone object $c_i \in \mathcal{C}^{\tau}$. In other words, C is obtained from C_i by composing with the canonical morphism $(c_i \to \operatorname{colim}_{\mathcal{J}} \phi \simeq c)$.

Let $\mathcal{J}_{\geq i}$ (resp. $\mathcal{J}_{>i}$) denote the full subcategory of $\mathcal{J}$ spanned by the objects $j \geq i$ (resp. $j > i$). Note that $\mathcal{J}_{\geq j}$ is again τ-filtered and $\mathcal{J}_{\geq i} \cong \Delta^0 \star \mathcal{J}_{>i}$. We have constructed a diagram in $\mathcal{C}$ as follows,

$$K \star \Delta^0 \cup_{\Delta^0} \Delta^0 \star \mathcal{J}_{>i} \xrightarrow{C_i \cup_{c_i} \phi_{|\mathcal{J}_{\geq i}}} \mathcal{C}^{\tau}.$$

The inclusion $K \star \Delta^0 \cup_{\Delta^0} \Delta^0 \star \mathcal{J}_{>i} \subseteq K \star \Delta^0 \star \mathcal{J}_{>i}$ is a Joyal equivalence [2, Lemma 5.4.5.10], therefore we obtain an extension of the last diagram:

$$K \star \Delta^0 \star \mathcal{J}_{>i} \to \mathcal{C}^{\tau}.$$

The adjoint of this map defines a diagram

$$\mathcal{J}_{\geq i} \to \mathcal{C}^{\tau}_{f/}.$$

We claim that C is canonically a colimit of this diagram. To see this, it suffices to observe that c is canonically a colimit of the composition $\mathcal{J}_{\geq i} \to \mathcal{C}^{\tau}_{f/} \to \mathcal{C}^{\tau} \to \mathcal{C}$, since $\mathcal{J}_{\geq i} \subseteq \mathcal{J}$ is cofinal. This completes the proof that $\mathcal{C}_{f/}$ is τ-accessible.

Since $\mathcal{C}^{\tau}_{f/} \subseteq (\mathcal{C}_{f/})^{\tau}$ generates $\mathcal{C}_{f/}$ under τ-filtered colimits, it follows that every τ-compact object in $\mathcal{C}_{f/}$ is a retract of an object in $\mathcal{C}^{\tau}_{f/}$. Since $\mathcal{C}^{\tau}_{f/}$ is idempotent complete, the required identification of $(\mathcal{C}_{f/})^{\tau}$ follows. □

For the proof of Claim (A), it will be convenient to work with the following weaker notion of *weak cofinality* (see also [2]). This notion will also be used for the proof of Claim (B) based on the observation that cofinality is a form of iterated weak cofinality.

Definition A.2. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then for any object $d \in \mathcal{D}$, we define the ∞-category $\mathcal{C}_{d/}$ by the pullback

$$\begin{tikzcd}
\mathcal{C}_{d/} \arrow[r] \arrow[d] & \mathcal{D}_{d/} \arrow[d] \\
\mathcal{C} \arrow[r, "p"'] & \mathcal{D}.
\end{tikzcd}$$

The functor p is *cofinal* if (and only if) for any object $d \in \mathcal{D}$, the ∞-category $\mathcal{C}_{d/}$ is weakly contractible. p is called *weakly cofinal* if for any object $d \in \mathcal{D}$, the ∞-category $\mathcal{C}_{d/}$ is non-empty.

Lemma A.3. *Assume the same notation as in Theorem 3.2.2.*

(1) *The functor* $p\colon \mathcal{C}^{\tau}_{/c} \to \mathcal{D}^{\tau}_{/d}$ *is weakly cofinal.*

(2) *Let* $f\colon K \to \mathcal{C}^{\tau}_{/c}$ *be a* τ*-small diagram. Then the induced functor*

$$(\mathcal{C}^{\tau}_{/c})_{f/} \to (\mathcal{D}^{\tau}_{/d})_{pf/}$$

is weakly cofinal.

(3) *Let* $h\colon T \star K \to (\mathcal{C}^{\tau}_{/c})$ *be a diagram where* T *is an arbitrary simplicial set and* K *is* τ*-small and weakly contractible. Then the induced functor*

$$(\mathcal{C}^{\tau}_{/c})_{h/} \to (\mathcal{D}^{\tau}_{/d})_{ph/}$$

is weakly cofinal.

Proof. (1): Let $(u\colon x \to d)$ be an object of $\mathcal{D}^{\tau}_{/d}$. We can write $c \simeq \operatorname{colim}_{\mathcal{J}} c_j$ as a τ-filtered colimit of τ-compact objects in $\mathcal{C}$. Since x is τ-compact in $\mathcal{D}$ and p preserves τ-filtered colimits, it follows that for some $(c_j \to c)$ there is a factorization in $\mathcal{D}$ of the form

$$\begin{tikzcd}
x \arrow[rr, "u"] \arrow[dr] & & p(c) = d. \\
& p(c_j) \arrow[ur] &
\end{tikzcd}$$

This shows that $(\mathcal{C}^{\tau}_{/c})_{u/}$ is non-empty, as required.

(2): Applying Proposition A.1, we obtain equivalences of ∞-categories:

$$(\mathcal{C}^{\tau}_{/c})_{f/} \simeq (\mathcal{C}^{\tau}_{f'/})_{/\widetilde{c}} \simeq (\mathcal{C}_{f'/})^{\tau}_{/\widetilde{c}}$$

$$(\mathcal{D}^{\tau}_{/d})_{pf/} \simeq (\mathcal{D}^{\tau}_{pf'/})_{/p\widetilde{c}} \simeq (\mathcal{D}_{pf'/})^{\tau}_{/p\widetilde{c}}$$

where $\widetilde{c} \in \mathcal{C}_{f'/}$ denotes the canonical cone on $f'\colon K \xrightarrow{f} \mathcal{C}^{\tau}_{/c} \to \mathcal{C}^{\tau}$ in $\mathcal{C}$ defined by the object $c \in \mathcal{C}$. Then the required result follows from (1) above for $\mathcal{C}_{f'/}$ in place of $\mathcal{C}$ and $\mathcal{D}_{pf'/}$ in place of $\mathcal{D}$.

(3): Let $i\colon K \hookrightarrow T \star K$ denote the inclusion and consider the commutative diagram:

$$\begin{tikzcd}
(\mathcal{C}^{\tau}_{/c})_{h/} \arrow[r] \arrow[d] & (\mathcal{D}^{\tau}_{/d})_{ph/} \arrow[d] \\
(\mathcal{C}^{\tau}_{/c})_{hi/} \arrow[r] & (\mathcal{D}^{\tau}_{/d})_{phi/}.
\end{tikzcd}$$

The inclusion i is right anodyne (see [2, Lemma 4.2.3.6]). It follows that the vertical maps are trivial fibrations. By (2), the lower horizontal functor is weakly cofinal. Therefore the upper horizontal functor is weakly cofinal as well. □

The following lemma proves Claim (A). Note that a proof of Claim (A) must show in particular that $\mathcal{C}''_{/c'}$ is non-empty – this turns out to be the main difficulty. The proof is based on a key construction which also highlights the interplay between the chosen regular cardinals κ and τ in Theorem 3.2.2.

Proposition A.4. *Assume the same notation as in Theorem 3.2.2. Let* $g\colon T \to \mathcal{C}''_{/c'}$ *be a diagram indexed by a* τ*-small simplicial set. Then the induced homotopy pullback*

$$\begin{tikzcd}
(\mathcal{C}''_{/c'})_{g/} \arrow[r] \arrow[d] & (\mathcal{C}^{\tau}_{/c})_{q'g/} \arrow[d, "p"] \\
(\mathcal{D}'^{\tau}_{/d'})_{p'g/} \arrow[r, "q"'] & (\mathcal{D}^{\tau}_{/d})_{pq'g/}
\end{tikzcd}$$

is non-empty. As a consequence, the ∞-category $\mathcal{C}''_{/c'}$ is τ-filtered.

Proof. See [2, Lemma 5.4.6.3]. (In the proof below, the symbol $\boxed{(*)}$ will indicate the arguments in the proof which use any of our special assumptions from Theorem 3.2.2.)

Let α be an ordinal. We consider the partially ordered sets $[\alpha] = \{\,\beta \colon \beta \preccurlyeq \alpha\,\}$ and $(\alpha) = \{\,\beta \colon \beta \prec \alpha\,\}$. Furthermore, we will call α even if $\alpha = \lambda + n$ where λ is a limit ordinal and n is even, and odd otherwise.

Let A be the partially ordered set of even ordinals which are smaller than κ and A' the partially ordered set of odd ordinals smaller than κ. Note that A and A' are κ-filtered, τ-small, and cofinal in (κ). We will construct a commutative diagram as follows:

$$\begin{tikzcd}
\mathrm{N}(A) \arrow[r] \arrow[d, "\overline{p}"'] & \mathrm{N}(\kappa) \arrow[d, "Q"] & \mathrm{N}(A') \arrow[l] \arrow[d, "\overline{q}"] \\
(\mathcal{C}^{\tau}_{/c})_{q'g/} \arrow[r, "p"'] & (\mathcal{D}^{\tau}_{/d})_{pq'g/} & (\mathcal{D}'^{\tau}_{/d'})_{p'g/}. \arrow[l, "q"]
\end{tikzcd}$$

Assuming that such a diagram has been constructed, we may then proceed as follows. First, since left fibrations create weakly contractible colimits [2, Proposition 4.4.2.9] and the lower ∞-categories define left fibrations over ∞-categories which admit τ-small κ-filtered colimits, it follows that the vertical diagrams $\overline{p}, Q$ and $\overline{q}$ $\boxed{(*)}$ admit colimits. Moreover, using [2, Proposition 4.4.2.9] and the fact that $p\colon \mathcal{C} \to \mathcal{D}$ and $q\colon \mathcal{D}' \to \mathcal{D}$ are κ-accessible, it follows that the lower horizontal functors in the $\boxed{(*)}$ diagram above (which are also denoted by p and q) preserve the colimits of the diagrams $\overline{p}$ and $\overline{q}$, respectively. Since the upper horizontal functors are cofinal, we have equivalences in $(\mathcal{D}^{\tau}_{/d})_{pq'g/}$

$$p(\operatorname{colim}_{\mathrm{N}(A)} \overline{p}) \simeq \operatorname{colim}_{\mathrm{N}(\kappa)} Q \simeq q(\operatorname{colim}_{\mathrm{N}(A')} \overline{q}).$$

These equivalences define an object in $(\mathcal{C}''_{/c'})_{g/}$, concluding the proof of the Proposition. Therefore it suffices to construct a commutative diagram as indicated above.

We will construct this diagram by transfinite induction. We will only treat the even case, since the other case is similar (but without the limit case). Suppose that $\overline{p}$ and Q have been constructed for all ordinals smaller than α. We will show how to extend these diagrams to $N(\{\beta \in A \colon \beta \preccurlyeq \alpha\})$ and $N([\alpha])$ respectively.

If α is a limit ordinal, then we can simply start by choosing a cone on $\overline{p}$

$$\begin{tikzcd}
\mathrm{N}(\{\,\beta \in A \colon \beta \prec \alpha\,\}) \arrow[r, "\overline{p}"] \arrow[d] & (\mathcal{C}^{\tau}_{/c})_{q'g/} \\
\mathrm{N}(\{\,\beta \in A \colon \beta \preccurlyeq \alpha\,\}) \arrow[ur, dashed, "\overline{p}^{\triangleright}"'] &
\end{tikzcd}$$

using the fact that the ∞-category in the target is τ-filtered – see the proof of $\boxed{(*)}$ Lemma A.3(2). Then we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{N}(\{\,\beta \in A \colon \beta \prec \alpha\,\}) \arrow[r] \arrow[d] & \mathrm{N}((\alpha)) \arrow[d, "Q"] \\
\mathrm{N}(\{\,\beta \in A \colon \beta \preccurlyeq \alpha\,\}) \arrow[r, "p \circ \overline{p}^{\triangleright}"'] & (\mathcal{D}^{\tau}_{/d})_{pq'g/}.
\end{tikzcd}$$

In order to extend Q to $N([\alpha])$, we have to solve the following lifting problem:

$$\begin{tikzcd}
\mathrm{N}(\{\,\beta \in A \colon \beta \preccurlyeq \alpha\,\}) \coprod_{\mathrm{N}(\{\,\beta \in A \colon \beta \prec \alpha\,\})} \mathrm{N}((\alpha)) \arrow[r] \arrow[d] & (\mathcal{D}^{\tau}_{/d})_{pq'g/}. \\
\mathrm{N}([\alpha]) \arrow[ur, dashed] &
\end{tikzcd}$$

Such an extension exists because the vertical monomorphism is a Joyal equivalence [2, Lemma 5.4.6.2]. This completes the inductive step in the case of a limit ordinal.

Now suppose that $\alpha = \alpha_0 + 1$ is a successor ordinal and assume that the diagrams $\overline{p}_{\prec\alpha} \colon \mathrm{N}(\{\,\beta \in A \colon \beta \prec \alpha\,\}) = \mathrm{N}(\{\,\beta \in A \colon \beta \prec \alpha_0\,\}) \to (\mathcal{C}^{\tau}_{/c})_{q'g/}$ and

$$Q \colon \mathrm{N}(\alpha) = \mathrm{N}(\{\,\beta \colon \beta \prec \alpha_0\,\} \cup \{\alpha_0\}) \to (\mathcal{D}^{\tau}_{/d})_{pq'g/}$$

have been constructed. By adjunction, these two compatible diagrams determine in particular an object

$$c(\alpha_0) \colon \Delta^0 \to ((\mathcal{D}^{\tau}_{/d})_{pq'g/})_{p\overline{p}_{\prec\alpha}/}.$$

By Lemma A.3, the canonical functor that is induced by p,

$$p_{\overline{p}_{\prec\alpha}/} \colon ((\mathcal{C}^{\tau}_{/c})_{q'g/})_{\overline{p}_{\prec\alpha}/} \to ((\mathcal{D}^{\tau}_{/d})_{pq'g/})_{p\overline{p}_{\prec\alpha}/}$$

$\boxed{(*)}$ is weakly cofinal. Therefore there is an object $c(\alpha) \colon \Delta^0 \to ((\mathcal{C}^{\tau}_{/c})_{q'g/})_{\overline{p}_{\prec\alpha}/}$ together with a morphism $\varphi \colon c(\alpha_0) \to p_{\overline{p}_{\prec\alpha}/}(c(\alpha))$. By adjunction, the objects $c(\alpha_0)$, $c(\alpha)$, and the morphism ϕ determine the dashed arrows in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{N}(\{\,\beta{\in}A \colon \beta{\preccurlyeq}\alpha\,\}) \arrow[r] \arrow[d, dashed] & \mathrm{N}(\{\,\beta{\in}A \colon \beta{\preccurlyeq}\alpha\,\}{\cup}\{\alpha_0\}) \arrow[d, dashed] & \mathrm{N}(\{\,\beta{\in}A \colon \beta{\prec}\alpha_0\,\}{\cup}\{\alpha_0\}) \arrow[l] \arrow[dl, dashed] \arrow[d] \\
(\mathcal{C}^{\tau}_{/c})_{q'g/} \arrow[r, "p"'] & (\mathcal{D}^{\tau}_{/d})_{pq'g/} & \mathrm{N}((\alpha)) \arrow[l, "Q"]
\end{tikzcd}$$

Thus, in order to extend Q to $N([\alpha])$, we need to solve the following lifting problem:

$$\begin{tikzcd}
\mathrm{N}(\{\,\beta \in A \colon \beta \preccurlyeq \alpha\,\} \cup \{\alpha_0\}) \coprod_{\mathrm{N}(\{\,\beta \in A \colon \beta \prec \alpha_0\,\} \cup \{\alpha_0\})} \mathrm{N}((\alpha)) \arrow[r] \arrow[d] & (\mathcal{D}^{\tau}_{/d})_{pq'g/} \\
\mathrm{N}([\alpha]) \arrow[ur, dashed] &
\end{tikzcd}$$

which is possible because the vertical monomorphism is a Joyal equivalence [2, Lemma 5.4.6.2]. This completes the inductive step in the case of a successor ordinal and concludes the proof of the Proposition. ☐

Lastly, the following proposition proves Claim (B).

Proposition A.5. *The functors* $p' \colon \mathcal{C}''_{/c'} \to \mathcal{D}'^{\tau}_{/d'}$ *and* $q' \colon \mathcal{C}''_{/c'} \to \mathcal{C}^{\tau}_{/c}$ *are cofinal.*

Proof. We only prove the claim in the case of q' since the proof for p' is completely analogous. Given an object $(x \to c) \in \mathcal{C}^{\tau}_{/c}$, we consider the commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] & \mathcal{C}''_{/c} \arrow[r] \arrow[d] & \mathcal{D}'^{\tau}_{/d'} \arrow[d] \\
(\mathcal{C}^{\tau}_{/c})_{(x\to c)/} \arrow[r, "\pi"'] & \mathcal{C}^{\tau}_{/c} \arrow[r] & \mathcal{D}^{\tau}_{/d}
\end{tikzcd}$$

in which both squares are (homotopy) pullbacks. It suffices to prove that $\mathcal{E}$ is filtered (and, as a consequence, weakly contractible). For any diagram $g\colon T \to \mathcal{E}$, where T is a finite simplicial set, we may consider the corresponding pullback diagrams that are obtained after slicing under g – similarly to Proposition A.4. We claim that the resulting pullback $\mathcal{E}_{g/}$ is non-empty.

This can be shown using essentially the same arguments as in the proof of Proposition A.4. Indeed, a close inspection of the proof of Proposition A.4 shows that it suffices to verify in addition the following properties (the essential requirements for the proof of Proposition A.4 are indicated by the symbol $\boxed{(*)}$ in the proof):

(1) For any diagram $u\colon T \to (\mathcal{C}^{\tau}_{/c})_{(x\to c)/}$ indexed by a finite simplicial set T, the ∞-category $((\mathcal{C}^{\tau}_{/c})_{(x\to c)/})_{u/}$ admits τ-small κ-filtered colimits. More generally, we note that the left fibration

$$((\mathcal{C}^{\tau}_{/c})_{(x\to c)/})_{u/} \to (\mathcal{C}^{\tau}_{/c})_{\pi u/}$$

creates weakly contractible colimits [2, Proposition 4.4.2.9].

(2) $(\mathcal{C}^{\tau}_{/c})_{(x\to c)/}$ is τ-filtered. This follows easily from the fact that $\mathcal{C}^{\tau}_{/c}$ is τ-filtered.

(3) For any diagram $v\colon T \star K \to (\mathcal{C}^{\tau}_{/c})_{(x\to c)/}$, where T is finite and K is τ-small and weakly contractible, the induced functor $((\mathcal{C}^{\tau}_{/c})_{(x\to c)/})_{v/} \to (\mathcal{C}^{\tau}_{/c})_{\pi v/}$ is weakly cofinal. This follows from Lemma A.3 for the ∞-category $(\mathcal{C}^{\tau}_{/c})_{(x\to c)/} \simeq (\mathcal{C}_{x/})^{\tau}_{/(x\to c)}$ in place of $\mathcal{C}^{\tau}_{/c}$ and $\mathcal{C}^{\tau}_{/c}$ in place of $\mathcal{D}^{\tau}_{/d}$.

□

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