FILTERED COLIMITS AND COMPACT OBJECTS

P. HANUKAEV

1. Filtered ∞-Categories

Definition 1.1. Let κ be a regular cardinal. We call a simplicial set X *κ-small* if the set of non-degenerate simplices of X has cardinality less than κ. If $\kappa = \omega$, then we also say that X is *finite*.

Remark 1.2. We call a category finite if both its set of objects and the set of all its morphisms are finite. An ordinary category $\mathcal{C}$ is filtered if and only if any diagram $I \to \mathcal{C}$ with I a finite category admits a cone $I^{\triangleright} \to \mathcal{C}$.

Definition 1.3. Let κ be a regular cardinal and $\mathcal{C}$ an ∞-category. We call $\mathcal{C}$ *κ-filtered* if for any κ-small simplicial set K and any diagram $f \colon K \to \mathcal{C}$ there exists $\overline{f} \colon K^{\triangleright} \to \mathcal{C}$ extending f. We say $\mathcal{C}$ is *filtered* if it is ω-filtered.

Remark 1.4. If $\mathcal{C}$ is an ordinary category, then $\mathcal{C}$ is filtered if and only if $\mathrm{N}(C)$ is filtered.

Remark 1.5. An ∞-category $\mathcal{C}$ is κ-filtered if and only if for any diagram $p \colon K \to \mathcal{C}$ with K a κ-small simplicial set, the slice ∞-category $\mathcal{C}_{p/}$ is non-empty. Any equivalence $f \colon \mathcal{C} \to \mathcal{D}$ induces an equivalence $\mathcal{C}_{p/} \to \mathcal{D}_{f\circ p/}$, so $\mathcal{C}$ is κ-filtered if and only if $\mathcal{D}$ is.

Proposition 1.6. *Let $\mathcal{C}$ a κ-filtered ∞-category. There exists a κ-filtered partially ordered set A and a cofinal map $\mathrm{N}(A) \to \mathcal{C}$.*

Proof. See [1, Proposition 5.3.1.18]. □

Lemma 1.7. *Any κ-filtered ∞-category is weakly contractible.*

Proof. Let $\mathcal{C}$ be a κ-filtered ∞-category. Since $\mathcal{C}$ is filtered it is non-empty so fix an object $C \in \mathcal{C}$ and write C also for the corresponding point of $|\mathcal{C}|$. We need to show, that for all $n \geq 0$, the set $\pi_n(|\mathcal{C}|, C)$ consists of a single point. Let $f \colon S^n \to |\mathcal{C}|$ be a map. There exists a finite simplicial subset $K \subseteq \mathcal{C}$ such that f factors through $|K| \subseteq |\mathcal{C}|$. Since $\mathcal{C}$ is filtered, the inclusion $K \hookrightarrow \mathcal{C}$ factors through $K^{\triangleright}$. Therefore, f factors as $S^n \to |K^{\triangleright}| \to |\mathcal{C}|$, which shows that f is null-homotopic, since $K^{\triangleright}$ is weakly contractible. □

Theorem 1.8. *Let $\mathcal{S}$ be the ∞-category of spaces, κ a regular cardinal and $\mathcal{I}$ an ∞-category. The following are equivalent:*

i) The ∞-category $\mathcal{I}$ is κ-filtered.
ii) The colimit functor $\mathrm{Fun}(\mathcal{I}, \mathcal{S}) \to \mathcal{S}$ preserves κ-small limits.

For a proof, see [1, Proposition 5.3.3.3].

2. Compact Objects

Definition 2.1. Let $\mathcal{C}$ be an ∞-category which admits small κ-filtered colimits.

i) A functor $f\colon \mathcal{C} \to \mathcal{D}$ is called *κ-continuous* if it preserves small κ-filtered colimits.
ii) Let $C \in \mathcal{C}$ be an object and let $j_C\colon \mathcal{C} \to \mathcal{S}$ be the functor corepresented by C. We say that C is κ-compact if j_C is κ-continuous.
iii) A left fibration $\mathcal{E} \to \mathcal{C}$ *κ-compact* if it corresponds to a κ-continuous functor $\mathcal{C} \to \mathcal{S}$.

Lemma 2.2. *Let κ be a regular cardinal.*

i) Let $\mathcal{C}$ be an ∞-category which admits small κ-filtered colimits and let $C \in \mathcal{C}$ be an object. The object C is κ-compact if and only if the left fibration $p\colon \mathcal{C}_{C/} \to \mathcal{C}$ is κ-compact.
ii) Let $f\colon \mathcal{C} \to \mathcal{D}$ be a κ-continuous functor, where $\mathcal{C}, \mathcal{D}$ are ∞-categories which admit small κ-filtered colimits. If $p\colon \mathcal{E} \to \mathcal{D}$ is a κ-compact left fibration, then $p'\colon \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \to \mathcal{C}$ is κ-compact.

Proof. See [1, Lemma 5.3.4.8].

i) p is classified by the corepresentable functor j_C.
ii) Note that p' is classified by the composite $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{\widetilde{p}} \mathcal{S}$ where $\widetilde{p}$ classifies p.

□

Lemma 2.3. *Let κ be a regular cardinal, $\mathcal{C}$ an ∞-category which admits small κ-filtered colimits and $f\colon c \to d$ a morphism in $\mathcal{C}$. If the objects c and d are κ-compact, then f is a κ-compact object of* $\mathrm{Fun}(\Delta^1, \mathcal{C})$.

Proof. See [1, Lemma 5.3.4.9]. Note that the corepresentable functor by f can be expressed as a finite limit of the corepresentable functors by c and d combined with the two projections $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$. □

Lemma 2.4. *Let κ be a regular cardinal and let $\{\mathcal{C}_\alpha\}$ be a κ-small family of ∞-categories with product $\mathcal{C}$. Suppose that each $\mathcal{C}_\alpha$ admits small κ-filtered colimits.*

i) The ∞-category $\mathcal{C}$ admits small κ-filtered colimits.
ii) If $C \in \mathcal{C}$ is an object such that the image of C in each $\mathcal{C}_\alpha$ is κ-compact, then C is a κ-compact object of $\mathcal{C}$.

Proof. See [1, Lemma 5.3.4.10]. (i) is obvious because colimits can be computed pointwise. (ii): Note that κ-small limits commute with κ-filtered colimits. □

Proposition 2.5. *Let κ be a regular cardinal, $\mathcal{C}$ an ∞-category which admits small κ-filtered colimits and $f\colon K \to \mathcal{C}$ a diagram with K a κ-small simplicial set. If $f(x)$ is κ-compact for each $x \in K$, then f is a κ-compact object of* $\mathrm{Fun}(K, \mathcal{C})$.

Proof. See [1, Proposition 5.3.4.13]. Call a simplicial set K *good* if it satisfies the conclusion of the Proposition. We want to show that any κ-small simplicial set is good.

(1) Suppose

$$\begin{array}{ccc} K' & \longrightarrow & K \\ \downarrow{\scriptstyle i} & & \downarrow \\ L' & \longrightarrow & L \end{array}$$

is a pushout square with K', K, L' good simplicial sets and i a cofibration. We claim that L is good: Consider the (homotopy) cartesian square

$$\begin{array}{ccc} \mathcal{C}^L & \longrightarrow & \mathcal{C}^{L'} \\ \downarrow & & \downarrow \\ \mathcal{C}^K & \longrightarrow & \mathcal{C}^{K'} \end{array}$$

All arrows in this square preserve κ-filtered colimits. If $f\colon L \to \mathcal{C}$ is as in the hypothesis, then the images of f in $\mathcal{C}^K$ and $\mathcal{C}^{L'}$ are κ-compact, so f is κ-compact (see [1, Lemma 5.4.5.7]).

(2) If $K \to K'$ is a Joyal-equivalence and K is good, then K' is good, because the restriction $\mathcal{C}^{K'} \to \mathcal{C}^K$ is an equivalence and hence preserves (and detects) κ-compact objects.

(3) For any $n \geq 0$ the simplex Δ^n is good. The inclusion

$$I^n = \Delta^1 \sqcup_{\Delta^0} \Delta^1 \sqcup_{\Delta^0} \ldots \sqcup_{\Delta^0} \Delta^1 \hookrightarrow \Delta^n$$

is a Joyal-equivalence. So, we only need to treat the case $n \leq 1$. The case $n = 0$ is clear, and the case $n = 1$ was Lemma 2.3.

(4) If $\{K_\alpha\}_\alpha$ is a κ-small collection of good simplicial sets and $K = \coprod_\alpha K_\alpha$, then K is good. We have $\mathrm{Fun}(K, \mathcal{C}) \cong \prod_\alpha \mathrm{Fun}(K_\alpha, \mathcal{C})$. If $f\colon K \to \mathcal{C}$ satisfies the hypothesis, then the image of f in each $\mathrm{Fun}(K_\alpha, \mathcal{C})$ satisfies the hypothesis and is therefore κ-compact. It follows from Lemma 2.4 that f is κ-compact.

(5) Any κ-small simplicial set K with $\dim K < \infty$ is good. By induction: The case $n = 0$ follows from parts (3) and (4). For $n > 0$ we have a pushout

$$\begin{array}{ccc} \coprod_\alpha \partial\Delta^n & \longrightarrow & \mathrm{sk}^{n-1} K \\ \downarrow & & \downarrow \\ \coprod_\alpha \Delta^n & \longrightarrow & K \end{array}$$

so K is good by the previous steps.

(6) Every κ-small simplicial set is good: If $\kappa = \omega$ this follows from (5), since ω-small means finite. Otherwise we have the skeletal filtration

$$\mathrm{sk}^0(K) \subseteq \mathrm{sk}^1(K) \subseteq \ldots$$

from which we obtain a tower of ∞-categories

$$\ldots \to \mathrm{Fun}(\mathrm{sk}^1(K), \mathcal{C}) \to \mathrm{Fun}(\mathrm{sk}^0(K), \mathcal{C})$$

with limit $\mathrm{Fun}(K, \mathcal{C})$. As in (1), each arrow in the tower preserves (κ-filtered) colimits. By a lemma similar to 2.4 (see [1, Lemma 5.3.4.12]) it suffices to check that the image of $f\colon K \to \mathcal{C}$ in each $\mathrm{Fun}(\mathrm{sk}^n(K), \mathcal{C})$ is κ-compact which holds by (5). $\square$

Corollary 2.6. *Let κ be a regular cardinal and $\mathcal{C}$ an ∞-category which has small κ-filtered colimits. If $p\colon K \to \mathcal{C}$ is a κ-small diagram such that for each $x \in K$ the object $p(x)$ of $\mathcal{C}$ is κ-compact, then the left fibration $\mathcal{C}_{p/} \to \mathcal{C}$ is κ-compact.*

Proof. See [1, Corollary 5.3.4.14]. $\square$

Corollary 2.7. *Let $\mathcal{C}$ be an ∞-category which has small κ-filtered colimits and let $\mathcal{C}^\kappa$ denote the full subcategory of κ-compact objects. Then $\mathcal{C}^\kappa$ is stable under formation of κ-small colimits which exist in $\mathcal{C}$.*

PROOF. Let K be a κ-small simplicial set, $p\colon K \to \mathcal{C}^\kappa$ a diagram and $\overline{p}\colon K^{\triangleright} \to \mathcal{C}$ a colimit cone over p. We have a diagram

$$\mathcal{C}_{p/} \longleftarrow \mathcal{C}_{\overline{p}/} \longrightarrow \mathcal{C}_{\overline{p}(\infty)/}$$

where the left map is a trivial fibration because $\overline{p}$ is a colimit and the right map is a trivial fibration because $\{\infty\} \to K^{\triangleright}$ is left anodyne. Hence, the left fibrations $\mathcal{C}_{p/} \to \mathcal{C}$ and $\mathcal{C}_{\overline{p}(\infty)/} \to \mathcal{C}$ are equivalent and the former is κ-compact, so $\overline{p}(\infty)$ is also κ-compact. $\square$

Remark 2.8. In the situation as above, $\mathcal{C}^\kappa$ is stable under retracts (see [1, Remark 5.3.4.16]).

Proposition 2.9. *Let $\mathcal{C}$ be a small ∞-category, κ a regular cardinal, and $C \in \mathcal{P}(\mathcal{C})$ an object. The following are equivalent:*

i) There exists a diagram $p\colon K \to \mathcal{C}$ with K a κ-small simplicial set such that $j \circ p$ has a colimit D in $\mathcal{P}(\mathcal{C})$ and C is a retract of D.
ii) The object C is κ-compact.

PROOF. We only present the proof of "(i) $\Rightarrow$ (ii)" and refer to [1, Proposition 5.3.4.17] for the converse.

Recall that we write $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ for the Yoneda embedding. By [1, Proposition 5.1.6.8], for any $A \in \mathcal{C}$ the object $j_A \in \mathcal{P}(\mathcal{C})$ is completely compact, that is the functor $\mathcal{P}(\mathcal{C}) \to \mathcal{S}$ corepresented by j_A preserves all small colimits. In particular the object j_A is κ-compact for any $A \in \mathcal{C}$. The claim now follows from Corollary 2.7 and Remark 2.8. $\square$

References

[1] Jacob Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press, 2009. Online revised version: `www.math.ias.edu/~lurie/papers/HTT.pdf`.