Dr. G. Raptis
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Universität Regensburg
Fakultät für Mathematik

Seminar:
Topics in higher category theory
(WS 19/20, Tuesday 16–18, M101)

1. Seminar Description

This is a seminar on advanced topics in the theory of ∞-categories. We will discuss the theory of several classes of ∞-categories which are especially important in higher category theory and its applications. More specifically, the seminar will focus on: accessible ∞-categories, presentable ∞-categories, ∞-topoi, and stable ∞-categories.

The main references for the seminar are Lurie's books [2, 3].

Prerequisites. Familiarity with the general theory of ∞-categories will be required (at the level of [2, Ch. 1–2, 4, and 5.2] or [1, Ch. 1–4 and 6]).

2. Schedule of Talks

Talk 1 (15.10.2019): *Introduction and overview.*

Talk 2 (22.10.2019): *∞-categories of presheaves.* Introduce the ∞-category of presheaves $\mathcal{P}(\mathcal{C})$ following [2, 5.1.0–5.1.5]. It might be helpful to recall the relationship between simplicial categories and ∞-categories (without proofs) [2, 1.1.5]. Some fundamental properties of $\mathcal{P}(\mathcal{C})$ should be explained, such as, for example, [2, Corollary 5.1.2.4]. Define the Yoneda embedding and prove the Yoneda lemma [2, 5.1.3]. State the universal property of $\mathcal{P}(\mathcal{C})$ [2, Theorem 5.1.5.6] and explain some of the ideas in the proof (focusing, especially, on the key [2, Lemma 5.1.5.3]). See also [1, Section 5.8] for an alternative approach.

Accessible ∞-categories

Talk 3 (29.10.2019): *Filtered colimits and compact objects.* The purpose of this talk is to discuss filtered colimits and prepare for the definition of the ∞-categories of Ind-objects $\mathrm{Ind}_\kappa(\mathcal{C})$. We will discuss some general properties of filtered ∞-categories, filtered colimits, and compact objects. There is a wealth of material on these topics in [2, 5.3.1–5.3.4], but we will focus on a soft selection of definitions and main results. Concerning filtered colimits, it should suffice to focus on [2, pp. 381–382], [2, Lemma 5.3.1.20], [2, Proposition 5.3.1.18] (without proof), and [2, 5.3.3.3]. Concerning compact objects, focus on the basic definitions and examples, and [2, 5.3.4.13–5.3.4.17].

Talk 4 (05.11.2019): *Ind–objects.* The goal of this talk is to introduce the ∞-categories of Ind-objects $\mathrm{Ind}_\kappa(\mathcal{C})$ and explain some of their properties following [2, 5.3.5]. Define $\mathrm{Ind}_\kappa(\mathcal{C})$ and mention the characterization in [2, Corollary 5.3.5.4] – which may be used as the definition. Explain the universal property of $\mathrm{Ind}_\kappa(\mathcal{C})$ in [2, Proposition 5.3.5.10] and [2, Proposition 5.3.5.11]. Discuss the relationship with the Yoneda embedding [2, Propositions 5.3.5.12 and 5.4.5.14]. Explain the functoriality of $\mathrm{Ind}_\kappa(-)$ following [2, Proposition 5.3.5.13, esp. (1) $\Rightarrow$ (2) $\Rightarrow$ (3)].

Talk 5 (12.11.2019): *Accessible ∞-categories.* Define accessible ∞-categories and explain some of their fundamental properties following [2, 5.4.1–5.4.6]. The necessary facts about locally small ∞-categories should be stated as needed (without proofs) [2, 5.4.1]. The main emphasis will be on the results of [2, 5.4.2], especially [2, Propositions 5.4.2.2 and 5.4.2.15]. Explain the main ideas for the accessibility of fiber products in [2, Proposition 5.4.6.6], emphasizing the results in [2, 5.4.5.2–5.4.5.5] which are also of independent interest.

Presentable ∞-categories

Talk 6 (19.11.2019): *Presentable ∞-categories: Structure theorems.* Introduce presentable ∞-categories following [2, 5.5.1–5.5.2]. Prove the characterization in [2, Theorem 5.5.1.1]. Then discuss [2, Corollary 5.5.2.4] as an application, and explain (at least one of) the adjoint functor theorems [2, Corollary 5.5.2.9]. See also [1, Section 7.11].

Talk 7 (26.11.2019): *Presentable ∞-categories: Localizations.* The goal of this talk is to classify accessible localizations of a presentable ∞-category following [2, 5.5.4–5.5.5]. General results about localization functors should be stated as needed (without proofs) [2, 5.2.7]. The main emphasis will be on explaining the construction of localizations [2, Proposition 5.5.4.15] and its universal property [2, Proposition 5.5.4.20]. If time permits, the construction of factorization systems [2, Proposition 5.5.5.7], which follows from [2, Lemma 5.5.5.14], should be sketched.

∞-topoi

Talk 8 (03.12.2019): *∞-topoi.* Define ∞-topoi and discuss their properties following [2, 6.1.0–6.1.5]. The main goal is to give an overview of the proof of [2, Theorem 6.1.0.6] (the more difficult implication (3) $\Rightarrow$ (1) will only be sketched). See also [6] and [5, Section 6] for an alternative approach, and [4, Sections 1–4].

No Seminar on 10.12.2019! (Windberg Meeting, 09.12.2019–11.12.2019)

Talk 9 (17.12.2019): *Sheaves and Grothendieck topologies.* The main goal of this talk is to explain the construction and properties of ∞-topoi of sheaves that arise from a Grothendieck topology [2, 6.2.1–6.2.4]. Introduce topological localizations and prove that they are accessible and left exact [2, Corollary 6.2.1.6]. Explain the notion of a Grothendieck topology for an ∞-category and then define the associated ∞-category of sheaves. Sketch the construction of the sheafification functor and conclude that this defines a topological localization [2, Proposition 6.2.2.7]. Moreover, show that every topological localzation arises in this way [2, Proposition 6.2.2.17]. If time permits, introduce effective epimorphisms and explain the universal property of the ∞-category of sheaves [2, Proposition 6.2.3.20]. See also [6] and [5, Section 11] for an alternative appraoch, and [4, Sections 5–8].

Talk 10 (07.01.2020): *∞-connectedness and hyperdescent.* Define homotopy groups in an ∞-topos and discuss the properties of n-connective maps following [2, 6.5.1] (esp. [2, Propositions 6.5.1.16 and 6.5.1.18]). Then define hypercomplete ∞-topoi and explain their properties following [2, 6.5.2]. The main emphasis will be on [2, 6.5.2.12–6.5.2.13] – the key fact [2, Proposition 6.5.2.8] may be used without proof. Introduce cotopological localizations and explain how together with topological localizations produce all the examples of ∞-topoi [2, 6.5.2.16–6.5.2.20]. Sketch the idea of the characterization of hypercompleteness in terms of hypercoverings [2, Theorem 6.5.3.12]. See also [6], [5, Sections 8–10], [4].

Talk 11 (14.01.2020): *The proper base change theorem.* Define proper maps between ∞-topoi and discuss their properties following [2, 7.3.1]. Then explain the relationship with proper maps of topological spaces [2, Theorem 7.3.1.16], giving an overview of the required results from [2, 7.3.2 and 7.3.4]. Then state and prove the nonabelian proper base change theorem [2, Corollary 7.3.1.18] and explain the connection with the abelian case [2, Remark 7.3.1.19].

Stable ∞-categories

Talk 12 (21.01.2020): *Stable ∞-categories.* Define stable ∞-categories and discuss some of their fundamental properties following [3, 1.1.1–1.1.4]. The property that the homotopy category of a stable ∞-category admits canonically the structure of a triangulated category should be explained [3, Theorem 1.1.2.14]. Discuss the closure properties of stable ∞-categories [3, 1.1.3] and the characterization in [3, Proposition 1.1.3.4]. Define exact functors [3, 1.1.4].

Talk 13 (28.01.2020): *Spectrum objects and stabilization.* Define the ∞-category of spectrum objects associated to an ∞-category with finite limits [3, 1.4.2]. Then prove that it is a stable ∞-category and that it satisfies a universal property [3, Proposition 1.4.2.16, Proposition 1.4.2.22, Proposition 1.4.2.24] – there are various interesting characterizations of stable ∞-categories in [3, 1.4.2] which should be mentioned. Discuss the example of the ∞-category of spectra (in spaces) [3, 1.4.3]. Explain the special properties of presentable stable ∞-categories following [3, 1.4.4], focusing especially on [3, 1.4.4.4–1.4.4.6 and 1.4.4.9].

Talk 14 (04.02.2020): *Differentiation.* Define the derivate of a functor and explain its properties following [3, 6.2.1]. In particular, the construction of derivatives [3, Proposition 6.2.1.9] and the chain rule [3, Theorem 6.2.1.22] should be explained. Moreover, if time permits, discuss the closely related notion of the differential of a functor [3, 6.2.3], focusing especially on [3, 6.2.3.13, 6.2.3.16–6.2.3.23].

References

[1] D.–C. Cisinski, *Higher Categories and Homotopical Algebra.* Cambridge studies in advanced mathematics 180, Cambridge University Press, 2019.
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[3] J. Lurie, *Higher algebra.* Available online: `http://www.math.harvard.edu/~lurie/`
[4] G. Raptis, *Seminar notes on ∞-topoi.* Available online: `https://graptismath.net/files/infty-topoi-notes.pdf`
[5] C. Rezk, *Toposes and homotopy toposes.* Available online: `https://faculty.math.illinois.edu/~rezk/homotopy-topos-sketch.pdf`
[6] B. Toën and G. Vezzosi, *Homotopical algebraic geometry. I. Topos theory.* Adv. Math. 193 (2005), no. 2, 257–372.