IND–OBJECTS

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1. Definition of $\mathrm{Ind}_\kappa(\mathcal{C})$ and basic properties

Let $\mathcal{C}$ be a small ∞-category and let $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ denote the Yoneda embedding. Given a presheaf $(f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}) \in \mathcal{P}(\mathcal{C})$, we define the **associated right fibration** by the pullback:

$$\begin{tikzcd} \widetilde{\mathcal{C}}(f) \colon = \mathcal{C}_{/f} \arrow[r] \arrow[d, "p(f)"'] & \mathcal{P}(\mathcal{C})_{/f} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{P}(\mathcal{C}). \end{tikzcd}$$

Definition 1. Let $\mathcal{C}$ be a small ∞-category and let κ be a regular cardinal. The ∞-category $\mathrm{Ind}_\kappa(\mathcal{C})$ is the full subcategory of $\mathcal{P}(\mathcal{C})$ which is spanned by those presheaves $(f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S})$ such that the ∞-category $\widetilde{\mathcal{C}}(f)$ is κ-filtered. We write $\mathrm{Ind}(\mathcal{C})$ when $\kappa = \omega$.

Example 2. Let $c \in \mathcal{C}$ be an object of $\mathcal{C}$ and let $j(c)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ be the associated representable presheaf. The associated right fibration can be identified with canonical right fibration $\mathcal{C}_{/c} \to \mathcal{C}$. The ∞-category $\mathcal{C}_{/c}$ has a terminal object and therefore is κ-filtered for any regular cardinal κ. It follows that $j(c) \in \mathrm{Ind}_\kappa(\mathcal{C})$ for any κ.

Proposition 3. *The full subcategory* $\mathrm{Ind}_\kappa(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ *is closed under κ-filtered colimits. In particular,* $\mathrm{Ind}_\kappa(\mathcal{C})$ *admits κ-filtered colimits.*

Proof. For every κ-filtered ∞-category K, there is a κ-filtered poset A and a cofinal functor $N(A) \to K$ [2, Proposition 5.3.1.18] . Therefore, by cofinality [2, Proposition 4.1.1.8], it suffices to prove that $\mathrm{Ind}_\kappa(\mathcal{C})$ is closed under κ-filtered colimits indexed by a κ-filtered poset A.

Let $\mathrm{Ind}'_\kappa(\mathcal{C})$ denote the full subcategory of (the ordinary category) $\mathrm{SSet}_{/\mathcal{C}}$ which is spanned by the right fibrations $\widetilde{\mathcal{C}} \to \mathcal{C}$ such that $\widetilde{\mathcal{C}}$ is κ-filtered. We note that $\mathrm{Ind}'_\kappa(\mathcal{C})$ is closed under κ-filtered colimits in $\mathrm{SSet}_{/\mathcal{C}}$, since the lifting or extension properties to decide whether an object is in $\mathrm{Ind}'_\kappa(\mathcal{C})$ are tested by κ-presentable objects. In addition, since the contravariant model category $\mathrm{SSet}_{/\mathcal{C}}$ is defined by a generating set of cofibrations between finitely presentable objects, it follows that κ-filtered colimits in $\mathrm{SSet}_{/\mathcal{C}}$ preserve weak equivalences (see [3, Proposition 4.1]). As a consequence, (ordinary) κ-filtered colimits in $\mathrm{SSet}_{/\mathcal{C}}$ compute homotopy κ-filtered colimits.

Homotopy (κ-filtered) colimits in $\mathrm{SSet}_{/\mathcal{C}}$ agree with (κ-filtered) colimits in the associated ∞-category $\mathrm{N}_\Delta\big((\mathrm{SSet}_{/\mathcal{C}})^\circ\big)$ [2, 4.2.4]. Using the equivalence of ∞-categories (see [2, Proposition 4.2.4.4]):

$$\mathrm{N}_\Delta(\mathrm{Fun}(A, \mathrm{SSet}_{/\mathcal{C}})^\circ) \simeq \mathrm{Fun}(N(A), \mathrm{N}_\Delta(\mathrm{SSet}^\circ_{/\mathcal{C}}))$$

(where the category on the left-hand side is endowed with the projective model structure), we conclude that

$$\mathrm{N}_\Delta\big((\mathrm{Ind}'_\kappa(\mathcal{C}))\big) \subseteq \mathrm{N}_\Delta\big((\mathrm{SSet}_{/\mathcal{C}})^\circ\big)$$

is closed under κ-filtered colimits. Then the result follows from the equivalences of ∞-categories $\mathrm{N}_\Delta\big((\mathrm{SSet}_{/\mathcal{C}})^\circ\big) \simeq \mathcal{P}(\mathcal{C})$ and $N_\Delta\big(\mathrm{Ind}'_\kappa(\mathcal{C})\big) \simeq \mathrm{Ind}_\kappa(\mathcal{C})$. □

Corollary 4. *Let $(f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}) \in \mathcal{P}(\mathcal{C})$. The following are equivalent:*

(1) There is a κ-filtered ∞-category K and a diagram $p\colon K \to \mathcal{C}$ such that f is a colimit of $K \xrightarrow{p} \mathcal{C} \xrightarrow{j} \mathcal{P}(\mathcal{C})$.

(2) $f \in \mathrm{Ind}_\kappa(\mathcal{C})$.

Proof. (2) ⇒ (1): The proof of the universal property of the Yoneda embedding shows that f is a colimit of the diagram:

$$\widetilde{\mathcal{C}}(f) \xrightarrow{p(f)} \mathcal{C} \xrightarrow{j} \mathcal{P}(\mathcal{C}).$$

(1) ⇒ (2): This follows from Proposition 3 and Example 2. □

Remark 5. Suppose that $\mathcal{C}$ has κ-small colimits. Then (1) and (2) of Corollary 4 are also equivalent to:

(3) f preserves κ-small limits.

See [2, Corollary 5.3.5.4]. As a consequence, assuming that $\mathcal{C}$ has κ-small colimits, $\mathrm{Ind}_\kappa(\mathcal{C})$ is exactly the full subcategory of $\mathcal{P}(\mathcal{C})$ that is spanned by those presheaves $f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ which preserve κ-small limits. See also [1, 1.42–1.45].

Remark 6. Let $c \in \mathcal{C}$ be an object of $\mathcal{C}$ and let $j(c) \in \mathrm{Ind}_\kappa(\mathcal{C})$ be the associated representable presheaf. Using the Yoneda lemma, we may identify the corepresentable functor by $j(c)$,

$$\mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{S},$$

with the functor

$$\mathrm{Ind}_\kappa(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C}) \xrightarrow{\mathrm{ev}_c} \mathcal{S}.$$

The last composite functor preserves κ-filtered colimits (by Proposition 3). Hence $j(c) \in \mathrm{Ind}_\kappa(\mathcal{C})$ is a κ-compact object.

Example 7. Let $\mathcal{C}$ be (the nerve of) an ordinary small category. Then $\mathrm{Ind}_\kappa(\mathcal{C})$ is equivalent to (the nerve of) an ordinary category.

2. Ind_κ-completion

Theorem 8. *Let $\mathcal{C}$ be a small ∞-category and let κ be a regular cardinal. Suppose that $\mathcal{D}$ is an ∞-category which admits κ-filtered colimits. Then the Yoneda embedding $j\colon \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ induces an equivalence of ∞-categories:*

$$\mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where $\mathrm{Fun}_\kappa(-,-)$ denotes the full subcategory of functors which preserve κ-filtered colimits.

Proof. Let $\mathcal{E}$ denote the opposite of the ∞-category of functors $\mathcal{D} \to \widehat{\mathcal{S}}$ where $\widehat{\mathcal{S}}$ is the ∞-category of (not necessarily small) spaces. Let $i\colon \mathcal{D} \to \mathcal{E}$ denote the opposite Yoneda embedding. The universal property of the Yoneda embedding $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ gives an equivalence

$$\mathrm{Fun}(\mathcal{C}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{E})$$

where $\mathrm{Fun}^L(-,-)$ denotes the full subcategory of functors which preserve small colimits. This functor is defined by considering left Kan extensions along the Yoneda embedding j. The composite functor

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C},\mathcal{D}) \arrow[r] \arrow[rrrd, dashed] & \mathrm{Fun}(\mathcal{C},\mathcal{E}) \arrow[r, "\simeq"] & \mathrm{Fun}^L(\mathcal{P}(\mathcal{C}),\mathcal{E}) \arrow[r, "\mathrm{res}"] & \mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(\mathcal{C}),\mathcal{E}) \\ & & & \mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(\mathcal{C}),\mathcal{D}) \arrow[u] \end{tikzcd}$$

factors through the full subcategory $\mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(\mathcal{C}),\mathcal{D}) \subseteq \mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(\mathcal{C}),\mathcal{E})$ because $i\colon \mathcal{D} \subseteq \mathcal{E}$ is closed under (κ-filtered) colimits. The dotted arrow defines an inverse to the restriction functor $\mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(\mathcal{C}),\mathcal{D}) \xrightarrow{j^*} \mathrm{Fun}(\mathcal{C},\mathcal{D})$. □

Proposition 9. *Let $\mathcal{C}$ be a small ∞-category and let κ be a regular cardinal. Suppose that $\mathcal{D}$ is an ∞-category which admits κ-filtered colimits and let*

$$F\colon \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{D}$$

be a functor which preserves κ-filtered colimits. Consider the composite functor $f\colon \mathcal{C} \xrightarrow{j} \mathrm{Ind}_\kappa(\mathcal{C}) \xrightarrow{F} \mathcal{D}$.

(1) If f is fully faithful and its essential image consists of κ-compact objects, then F is fully faithful.
(2) F is an equivalence if and only if:
  (i) f is fully faithful.
  (ii) f factors through $\mathcal{D}^\kappa$.
  (iii) $\{f(c)\}_{c\in\mathcal{C}^\kappa}$ generate $\mathcal{D}$ under κ-filtered colimits.

Proof. (1): We need to show that for every $x, y \in \mathrm{Ind}_\kappa(\mathcal{C})$, the canonical map induced by F:

$$(*) \qquad \mathrm{map}_{\mathcal{P}(\mathcal{C})}(x,y) \to \mathrm{map}_{\mathcal{D}}(F(x),F(y))$$

is an equivalence. This holds when $x, y \in \mathcal{C} \subseteq \mathrm{Ind}_\kappa(\mathcal{C})$, by assumption. Since every $y \in \mathrm{Ind}_\kappa(\mathcal{C})$ is a κ-filtered colimit of representables and $x \in \mathcal{C} \subseteq \mathrm{Ind}_\kappa(\mathcal{C})$ is κ-compact (Remark 6), it follows using the assumptions on F that (*) is an equivalence whenever $x \in \mathcal{C} \subseteq \mathrm{Ind}_\kappa(\mathcal{C})$. Since every $x \in \mathrm{Ind}_\kappa(\mathcal{C})$ is a (κ-filtered) colimit of representables, the assertion then holds also for any $x \in \mathrm{Ind}_\kappa(\mathcal{C})$.

(2): Suppose that F is an equivalence. Then (i) follows from the Yoneda lemma, (ii) follows from Remark 6, and (iii) is a consequence of Corollary 4. Conversely, if (i)–(iii) are satisfied, then F is fully faithful (by (1)) and essential surjective (by (i) and Corollary 4). □

Example 10. Let $\mathcal{C}$ be a small ∞-category and let κ be a regular cardinal. The ∞-category $\mathcal{P}(\mathcal{C})^\kappa$ is essentially small (as a consequence of the fact that every κ-compact object is a retract of a κ-small colimit of representable objects [2, Proposition 5.3.4.17].) Then Proposition 9 shows that the inclusion functor $\mathcal{P}(\mathcal{C})^\kappa \subseteq \mathcal{P}(\mathcal{C})$ extends to an equivalence:

$$\mathrm{Ind}_\kappa\big(\mathcal{P}(\mathcal{C})^\kappa\big) \xrightarrow{\simeq} \mathcal{P}(\mathcal{C}).$$

Remark 11. The functor $j\colon \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ preserves κ-small colimits (which exist in $\mathcal{C}$). To see this, consider a diagram $K \to \mathcal{C}$, $i \mapsto c_i$, where K is κ-small and let $c \in \mathcal{C}$ be (the cone-object of) a colimit of F. Then using the Yoneda lemma, Proposition

3, Corollary 4 and the fact that κ-filtered colimits commute with κ-small limits in $\mathcal{S}$, we have canonical equivalences of mapping spaces for every $G \in \mathrm{Ind}_\kappa(\mathcal{C})$:

$$\begin{aligned} \mathrm{map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(j(c), G) &\simeq \mathrm{map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(j(c), \mathrm{colim}_{\beta \in J} G_\beta) \\ &\simeq (\mathrm{colim}_{\beta \in J} G_\beta)(c) \simeq \mathrm{colim}_{\beta \in J} G_\beta(c) \\ &\simeq \mathrm{colim}_{\beta \in J} \lim_{i \in K^{\mathrm{op}}} G_\beta(c_i) \\ &\simeq \lim_{i \in K^{\mathrm{op}}} \mathrm{colim}_{\beta \in J} G_\beta(c_i) \\ &\simeq \lim_{i \in K^{\mathrm{op}}} G(c_i) \simeq \lim_{i \in K^{\mathrm{op}}} \mathrm{map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(j(c_i), G) \end{aligned}$$

where the diagram $J \to \mathrm{Ind}_\kappa(\mathcal{C})$, $\beta \mapsto G_\beta$, is a κ-filtered diagram of representables with colimit G. This shows that $j(c)$ is (the cone-object of) a colimit of the composition $K \to \mathcal{C} \xrightarrow{j} \mathrm{Ind}_\kappa(\mathcal{C})$. See also [2, Proposition 5.3.5.14].

3. Functoriality

Let $f\colon \mathcal{C} \to \mathcal{C}'$ be a functor between small ∞-categories. Applying Theorem 8, we obtain an extension of the composite functor

$$\mathcal{C} \xrightarrow{f} \mathcal{C}' \xrightarrow{j'} \mathrm{Ind}_\kappa(\mathcal{C}')$$

to a canonical functor

$$\mathrm{Ind}_\kappa(f)\colon \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathrm{Ind}_\kappa(\mathcal{C}')$$

which preserves κ-filtered colimits.

Proposition 12. *Let $f\colon \mathcal{C} \to \mathcal{C}'$ be a functor between small ∞-categories. The following are equivalent:*

(1) The restriction functor $f^*\colon \mathcal{P}(\mathcal{C}') \to \mathcal{P}(\mathcal{C})$, which is given by precomposition with f, restricts to a functor $\mathrm{Ind}_\kappa(\mathcal{C}') \to \mathrm{Ind}_\kappa(\mathcal{C})$.

(2) $\mathrm{Ind}_\kappa(f)$ admits a right adjoint.

Proof. (1) $\Rightarrow$ (2): Let $f_!\colon \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}')$ denote the left adjoint of f^* – this is the unique functor which extends f and preserves small colimits. The restriction of $f_!$ to $\mathrm{Ind}_\kappa(\mathcal{C})$ is $\mathrm{Ind}_\kappa(f)$. Then the adjoint pair $(f_!, f^*)$ restricts to an adjoint pair between the Ind-categories.

(2) $\Rightarrow$ (1): See [2, Proposition 5.3.5.13]. □

References

[1] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories.* London Mathematical Society Lecture Note Series 189, Cambridge University Press, 1994.

[2] J. Lurie. *Higher topos theory.* Annals of Mathematics Studies, Vol. 170. Princeton University Press, Princeton, NJ, 2009. Online revised version: `www.math.harvard.edu/~lurie/papers/highertopoi.pdf`

[3] G. Raptis and J. Rosický. *The accessibility rank of weak equivalences.* Theory Appl. Categ. 30 (2015), no. 19, 687–703.