∞-CATEGORIES OF PRESHEAVES

BENEDIKT PREIS

1. Introduction

Our goal is to define the Yoneda embedding for ∞-categories and prove that it satisfies a universal property.

Let us first recall the 1-categorical context. Let $\mathcal{C}$ be an ordinary (small) category and let $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, Sets)$ denote the (ordinary) category of presheaves on $\mathcal{C}$. The *Yoneda-functor*

$$y : \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{C}), \quad c \mapsto \mathrm{Hom}_{\mathcal{C}}(\bullet, c).$$

is fully faithful and satisfies the following universal property:

Theorem 1.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories where $\mathcal{C}$ is small and $\mathcal{D}$ admits all small colimits. Then there exists a unique (up to unique natural isomorphism) colimit-preserving functor $F_! : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ such that the diagram:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle y}\downarrow & \nearrow_{F_!} & \\ \mathcal{P}(\mathcal{C}) & & \end{array}$$

commutes up to a natural isomorphism. Moreover, restriction along the Yoneda embedding induces an equivalence of ∞-categories,

$$y^* : \mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{D})$ which is spanned by colimit-preserving functors.

The functor $F_!$ is the left Kan-extension of the functor $F : \mathcal{C} \to \mathcal{D}$ along the Yoneda embedding y. The main idea of the proof of Theorem 1.1 is that for any presheaf $X \in \mathcal{P}(\mathcal{C})$, we have a canonical identification in $\mathcal{P}(\mathcal{C})$:

$$X \cong \mathrm{colim}_{y(c)\to X}\, y(c)$$

which can be used to identify $F_!(X)$ canonically with $\mathrm{colim}_{y(c)\to X}\, F(c)$. Moreover, $F_!$ has a right adjoint

$$F^* : \mathcal{D} \to \mathcal{P}(\mathcal{C})$$

given on objects by

$$d \mapsto \mathrm{Hom}_{\mathcal{D}}(F(\bullet), d).$$

Examples of adjunctions of this form include:

(1) The (homotopy category/nerve functor)-adjunction between simplicial sets and small categories (see [3, 1.2.3]):

$$ho \dashv N : \mathrm{sSet} \rightleftarrows \mathrm{Cat}\,.$$

(2) The (geometric realization/singular set)-adjunction between simplicial sets and topological spaces:

$$|-| \dashv \mathrm{Sing}(-) : \mathrm{sSet} \rightleftarrows \mathrm{Top}\,.$$

2. Simplicial categories and the Coherent Nerve Functor

Let Cat_Δ denote the category of small simplicially enriched categories. There is an adjunction:

$$(*) \qquad \mathrm{sSet} \underset{N_\Delta}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathrm{Cat}_\Delta\,, \quad \mathfrak{C} \dashv N_\Delta$$

arising as the extension by colimits of a functor/cosimplicial object $\mathfrak{C} : \Delta \longrightarrow \mathrm{Cat}_\Delta$. This functor is defined as follows:

(i) $\mathrm{Ob}\,\mathfrak{C}([n]) = \{0, \dots, n\}$

(ii)

$$\mathrm{Hom}_{\mathfrak{C}([n])}(i,j) = \begin{cases} \varnothing & \text{if } i > j \\ N(P_{i,j}) & \text{if } i \leq j \end{cases}$$

where $P_{i,j}$ is the poset of subsets

$$P_{i,j} = \{I \subset [i,j] \,|\, i,j \in I\}$$

partially ordered by inclusion of subsets.

(iii) The composition in $\mathfrak{C}([n])$ is induced by the operation of taking unions of subsets.

The right adjoint N_Δ is the *coherent nerve functor*.

Let us describe the adjunction (*) in some more detail. Given a simplicial category S, we obtain a simplicial set $N_\Delta(S)$ defined by

$$N_\Delta(S)_n = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}([n]), S).$$

The lower degrees of the cosimplicial object $\mathfrak{C}([\bullet])$ can be described as follows:

(0) $\mathfrak{C}([0])$ has one object 0 and $\mathrm{Hom}_{\mathfrak{C}([0])}(0,0)$ is isomorphic to Δ^0.

(1) $\mathfrak{C}([1])$ has two objects 0 and 1 and every Hom-sSet is isomorphic to Δ^0 except for $\mathrm{Hom}_{\mathfrak{C}([1])}(1,0)$, which is empty. In other words, it is isomorphic to the (simplicial) category Δ^1.

(2) $\mathfrak{C}([2])$ consists of three objects $0, 1$ and 2 such that

$$\mathrm{Hom}_{\mathfrak{C}([2])}(i,i)\ (0 \leq i \leq 2),\ \mathrm{Hom}_{\mathfrak{C}([2])}(0,1),\ \text{and}\ \mathrm{Hom}_{\mathfrak{C}([2])}(1,2)$$

are isomorphic to Δ^0,

$$\mathrm{Hom}_{\mathfrak{C}([2])}(0,2) = N(\{0,2\} < \{0,1,2\}) \cong \Delta^1,$$

and every other Hom-sSet is empty.

We can then directly describe the 0-,1- and 2-simplicies of $N_\Delta(S)$ as follows:

(0) $N_\Delta(S)_0 =$ Ob S.

(1) An element in $N_\Delta(S)_1$ is specified by two objects $x, y \in S$ and a 0-simplex in $\mathrm{Hom}_S(x,y)_0$.

(2) An element of $N_\Delta(S)_2$ is specified by three objects $x, y, z \in S$, 0-simplices $f \in \mathrm{Hom}_S(x,y)_0, g \in \mathrm{Hom}_S(y,z)_0$ and $h \in \mathrm{Hom}_S(x,z)_0$ together with an 1-simplex (= path) $h \to g \circ f$ in $\mathrm{Hom}_S(x,z)_1$.

See also [3, 1.1.5].

Lemma 2.1. *Let $\mathcal{D}$ be an ∞-category. Then there is a natural isomorphism* $\operatorname{Ob}\mathfrak{C}(\mathcal{D}) \cong \mathcal{D}_0$.

Proof. Both functors $\operatorname{Ob}\mathfrak{C}(-), (-)_0 : \mathrm{sSet} \to \mathrm{Set}$ preserve colimits. Moreover, using the description of $\mathfrak{C}$ above, their restrictions along the Yoneda embedding $y : \Delta \to \mathrm{sSet}$, $[n] \mapsto \Delta^n$, are canonically isomorphic. □

Proposition 2.2. *Let $\mathcal{C}$ be a simplicial category such that* $\operatorname{Hom}_{\mathcal{C}}(x, y)$ *is a Kan complex for every $x, y \in \mathcal{C}$. Then the coherent nerve $N_\Delta(\mathcal{C})$ is an $\infty-$category.*

Proof. See [3, Proposition 1.1.5.10]. □

Definition 2.3. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between simplicial categories in Cat_Δ. We call F a *categorical equivalence* if the following are satisfied:

(1) F is *weakly fully faithful*, i.e., it induces weak equivalences $F : \operatorname{Hom}_{\mathcal{C}}(x, y) \to \operatorname{Hom}_{\mathcal{D}}(Fx, Fy)$ for every $x, y \in \mathcal{C}$.
(2) The functor $\pi(F)$ is essentially surjective. Here $\pi : \mathrm{Cat}_\Delta \to \mathrm{Cat}$ is induced by the monoidal functor $\pi_0 : \mathrm{sSet} \to \mathrm{Set}$.

There is a model structure on Cat_Δ where the weak equivalences are the categorical equivalences and the fibrations are the simplicial functors $f : \mathcal{C} \to \mathcal{D}$ satisfying:

(1) The map $f : \operatorname{Hom}_{\mathcal{C}}(x, y) \to \operatorname{Hom}_{\mathcal{D}}(fx, fy)$ is a Kan fibration for every $x, y \in \mathcal{C}$.
(2) For any $y \in \mathcal{D}$, $x \in \mathcal{C}$ and a homotopy equivalence $a : y \to f(x)$, that is, a morphism which becomes an isomorphism in $\pi(\mathcal{D})$, there exists $x' \in \mathcal{C}$ and a homotopy equivalence $b : x' \to x$ such that $f(b) = a$.

See [1] and [3, A.3.2]. With this model structure and the Joyal model structure on sSet, the adjunction (*) becomes a Quillen equivalence (see, for example, [3, 2.2]). In particular, this means that for every fibrant simplicial category $\mathcal{D}$ in Cat_Δ, the canonical counit map

$$\epsilon_{\mathcal{D}} : \mathfrak{C}N_\Delta(\mathcal{D}) \to \mathcal{D}$$

is a categorical equivalence. The following proposition is an easy consequence of the Quillen equivalence $(\mathfrak{C}, N_\Delta)$.

Proposition 2.4. *Let $\mathcal{C}$ be an ∞-category, $\mathcal{D}$ a fibrant simplicial category in* Cat_Δ*, and let $f : \mathcal{C} \to N_\Delta(\mathcal{D})$ be a functor of ∞-categories. We denote by $\tilde{f} : \mathfrak{C}(\mathcal{C}) \to \mathcal{D}$ the morphism in* Cat_Δ *which is adjoint to f. Then $\tilde{f}$ is weakly fully faithful if and only if f is fully faithful.*

Proof. Let $i : \mathcal{E} \hookrightarrow N_\Delta(\mathcal{D})$ denote the full subcategory spanned by the objects in the image of f. Consider the factorization of $\tilde{f}$

$$\begin{array}{ccccc}
\mathfrak{C}(\mathcal{C}) & \xrightarrow{\mathfrak{C}(f)} & \mathfrak{C}(N_\Delta(\mathcal{D})) & \xrightarrow{\epsilon_{\mathcal{D}}} & \mathcal{D} \\
& \searrow^{\mathfrak{C}(f')} & \uparrow{\scriptstyle \mathfrak{C}(i)} & & \\
& & \mathfrak{C}(\mathcal{E}) & &
\end{array}$$

where $\epsilon_{\mathcal{D}}$ is a categorical equivalence. It follows from the definition of $\mathfrak{C}(-)$ that the functor $\mathfrak{C}(i)$ is weakly fully faithful. Hence $\tilde{f}$ is weakly fully faithful if and only if $\mathfrak{C}(f')$ is a categorical equivalence. Using the Quillen equivalence $(\mathfrak{C}, N_\Delta)$, $\mathfrak{C}(f')$

is a categorical equivalence if and only if f' is a Joyal equivalence ($\iff$ f is fully faithful). $\square$

3. Presheaves

Let Kan denote the full simplicical subcategory of sSet consisting of Kan complexes with the usual simplicial enrichment (Kan is enriched in Kan complexes), and let $\mathcal{S} := N_\Delta(\mathrm{Kan})$ be (a model for) the ∞-category of spaces.

Definition 3.1. Let S be a simplicial set. We define the *∞-category $\mathcal{P}(S)$ of presheaves on S* to be $\mathrm{Fun}(S^{op}, \mathcal{S})$.

An equivalent model for the ∞-category of presheaves on S is constructed as follows. Start with $\mathcal{C} \in \mathrm{Cat}_\Delta$ such that $S \simeq N_\Delta(\mathcal{C})$. We consider the simplical category of simplicial functors $\mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathrm{sSet})$ equipped with the projective model structure where the weak equivalences and the fibrations are defined pointwise (with respect to the Kan–Quillen model structure on sSet). There are categorical equivalences of simplicial categories,

$$(**) \qquad \mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathrm{sSet})^\circ \simeq \mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathrm{sSet}^\circ) \cong \mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathrm{Kan}),$$

and $\mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathrm{sSet})^\circ$ (= full subcategory of fibrant–cofibrant objects in the simplicial model category $\mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathrm{sSet})$) is enriched in Kan complexes. Then

$$\mathcal{P}'(S) := N_\Delta(\mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathrm{sSet})^\circ)$$

is an ∞-category. As a consequence of general deep results on the comparison between the homotopy theories of strict and homotopy coherent diagrams [3, Proposition 4.2.4.4], there is a natural equivalence of ∞-categories

$$\mathcal{P}'(S) \simeq \mathcal{P}(S).$$

See [3, 5.1.1]. We recall that this equivalence is obtained from the canonical evaluation map

$$\mathrm{Fun}_\Delta(\mathcal{C}^{op}, \mathrm{Kan}) \times \mathcal{C}^{op} \to \mathrm{Kan}\,.$$

after applying N_Δ, passing to the adjoint map, and using (**).

Construction 3.2. We give a construction of the Yoneda-functor $\mathrm{y}_S : S \to \mathcal{P}(S)$ in the ∞-categorical context following [3]. Let S be a simplicial set. For $x, y \in \mathfrak{C}(S)$, the rule

$$(x, y) \mapsto \mathrm{Sing}\,|\mathrm{Hom}_{\mathfrak{C}(S)}(x, y)|$$

determines a simplicial functor

$$\widehat{\mathrm{Hom}} : \mathfrak{C}(S) \times \mathfrak{C}(S)^{op} \longrightarrow \mathrm{Kan}\,.$$

Thus, we obtain a composite simplicial functor

$$\mathfrak{C}(S \times S^{op}) \longrightarrow \mathfrak{C}(S) \times \mathfrak{C}(S)^{op} \xrightarrow{\widehat{\mathrm{Hom}}} \mathrm{Kan}\,.$$

By adjunction, this induces a map of simplicial sets

$$S \times S^{op} \longrightarrow N_\Delta(\mathrm{Kan}) \simeq \mathcal{S}$$

which in turn defines by adjunction the required Yoneda-functor

$$\mathrm{y}_S : S \longrightarrow \mathcal{P}(S).$$

In the case where S is an ∞-category, we have for any $x, y \in S$

$$\mathrm{y}_S(x)(y) = \mathrm{Sing}\,|\mathrm{Hom}_{\mathfrak{C}(S)}(X, Y)| \simeq \mathrm{map}_S(x, y)$$

where the last equivalence is shown in [3, 2.2.4].

Proposition 3.3 (∞-categorical Yoneda embedding)**.** *The Yoneda-functor* $\mathrm{y}_S : S \to \mathcal{P}(S)$ *is fully faithful for every simplicial set* S*.*

Proof. See [3, Proposition 5.1.3.1]. Let $\mathcal{C} = \mathrm{Sing}|\mathfrak{C}(S^{op})|$ be a fibrant replacement of the simplicial category $\mathfrak{C}(S^{op})$, given by replacing $\mathrm{Hom}_{\mathfrak{C}(S^{op})}(x,y)$ with $\mathrm{Sing}|\mathrm{Hom}_{\mathfrak{C}(S^{op})}(x,y)|$ for all $x,y \in \mathfrak{C}(S^{op})$. Let $\widetilde{y}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}_{\Delta}(\mathcal{C}^{op}, \mathrm{sSet})$ denote the usual (enriched) Yoneda embedding of simplicial categories – which is fully faithful in the ordinary (enriched) sense. We observe that y_S factors as

$$S \xrightarrow{a} N_{\Delta}\big(\mathrm{Fun}_{\Delta}(\mathcal{C}^{op}, \mathrm{Kan})\big) \xrightarrow{b} \mathrm{Fun}(S^{op}, \mathcal{S}),$$

where a is the adjoint of the map

$$a' : \mathfrak{C}(S) \xrightarrow{\simeq} \mathcal{C} \xrightarrow{\widetilde{y}_{\mathcal{C}}} \mathrm{Fun}_{\Delta}(\mathcal{C}^{op}, \mathrm{Kan})$$

and b is the equivalence of ∞-categories (that was discussed above):

$$N_{\Delta}\big(\mathrm{Fun}_{\Delta}(\mathcal{C}^{op}, \mathrm{Kan})\big) \simeq N_{\Delta}\big(\mathrm{Fun}_{\Delta}(\mathfrak{C}(S^{op}), \mathrm{Kan})\big) \simeq \mathrm{Fun}(S^{op}, \mathcal{S}).$$

Hence it is enough to show that a is fully faithful. By Proposition 2.4, it suffices to show that a' is (weakly) fully faithful, which follows immediately from the fact that the (enriched) Yoneda embedding $\widetilde{y}_{\mathcal{C}}$ is fully faithful. □

4. The universal property of the Yoneda embedding

Proposition 4.1. *Let* S *be a simplicial set. The* ∞*-category* $\mathcal{P}(S)$ *of presheaves on* S *admits all small limits and colimits and these can be computed pointwise. (This means that a cone* $\overline{F} : K^{\triangleright} \to \mathcal{P}(S)$ *is a colimit cone if and only if* $\mathrm{ev}_x \circ \overline{F} : K^{\triangleright} \to \mathcal{S}$ *is a colimit cone for every* $x \in S$ *– and similarly for limits.)*

Proof. See [3, 5.1.2]. We will only sketch the idea of the proof in the case of colimits. Limits can be treated using similar arguments. The proof is an easy consequence of the following facts:

(a) The ∞-category $\mathcal{S}$ of spaces admits all small colimits (and limits). (See, for example, the discussion in [3, 5.1.2].)

(b) (see [2, 6.2]) For any ∞-category $\mathcal{C}$ and any simplicial set K, $\mathcal{C}$ admits K-shaped colimits if and only if the constant diagram functor

$$\mathrm{const}_K : \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$$

admits a left adjoint. In this case, the left adjoint is given by

$$\mathrm{colim}_K : \mathrm{Fun}(K, \mathcal{C}) \to \mathcal{C}$$
$$F \mapsto \mathrm{colim}_K F.$$

Combining (a) and (b), we conclude that $\mathrm{const}_{S^{op}} : \mathcal{S} \to \mathcal{P}(S)$ has a left adjoint $\mathrm{colim}_{S^{op}}$ for any simplicial set S. It follows that for any simplicial set K, the constant diagram functor of K-diagrams in $\mathcal{P}(S)$:

$$\mathcal{P}(S) = \mathrm{Fun}(S^{op}, \mathcal{S}) \xrightarrow{(\mathrm{const}_K)_*} \mathrm{Fun}(S^{op}, \mathrm{Fun}(K, \mathcal{S})) \cong \mathrm{Fun}(K, \mathrm{Fun}(S^{op}, \mathcal{S}))$$

has a left adjoint, given pointwise as follows,

$$\mathrm{Fun}(K, \mathrm{Fun}(S^{op}, \mathcal{S})) \cong \mathrm{Fun}(S^{op}, \mathrm{Fun}(K, \mathcal{S})) \xrightarrow{(\mathrm{colim}_K)_*} \mathrm{Fun}(S^{op}, \mathcal{S}).$$

□

The main result of this section is the following ∞-categorical analogue of Theorem 1.1.

Theorem 4.2 (Universal property of the Yoneda embedding)**.** *Let $\mathcal{C}$ be a small ∞-category and $\mathcal{D}$ an ∞-category which admits small colimits. Then the restriction functor induced by the Yoneda embedding*

$$\mathrm{y}_{\mathcal{C}}^* : \mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is an equivalence of ∞-categories, where $\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{D})$ which is spanned by those functors which preserve small colimits.

See [3, Theorem 5.1.5.6]. The main ingredient of the proof, analogously to the 1-categorical case, is the following:

Proposition 4.3. *Let $\mathcal{C}$ be a small ∞-category and $F \in \mathcal{P}(\mathcal{C})$. Then F is the colimit of the canonical diagram:*

$$\phi_F : \mathcal{C}_{/F} = \mathcal{C} \times_{\mathcal{P}(\mathcal{C})} \mathcal{P}(\mathcal{C})_{/F} \to \mathcal{P}(\mathcal{C})_{/F} \longrightarrow \mathcal{P}(\mathcal{C}).$$

Proof. See [3, Lemma 5.1.5.3]. By Proposition 4.1, it is enough to show that for any $x \in \mathcal{C}$, $F(x)$ is the colimit of the composition of ϕ_F with the evaluation map

$$\mathrm{ev}_x \circ \phi_F : \mathcal{C}_{/F} \subset \mathcal{P}(\mathcal{C})_{/F} \longrightarrow \mathcal{P}(\mathcal{C}) \xrightarrow{ev_x} \mathcal{S}\,.$$

Indeed we have a chain of natural equivalences:

$$\begin{aligned}
\mathrm{colim}_{\mathcal{C}_{/F}}(\mathrm{ev}_x \circ \phi_F) \simeq \mathrm{colim}_{\mathrm{y}_{\mathcal{C}}(c)\to F}\, \mathrm{map}_{\mathcal{C}}(x, c) &\simeq \mathrm{colim}_{y_{\mathcal{C}}(c)\to F}\, \mathrm{colim}_{x \to c} *\\
(J = \mathcal{C}_{/F} \times_{\mathcal{C}} \mathcal{C}_{x/}) \qquad\qquad &\overset{(1)}{\simeq} \mathrm{colim}_J *\\
(I = \mathcal{C}_{/F} \times_{\mathcal{C}} \{x\}) \qquad\qquad &\overset{(2)}{\simeq} \mathrm{colim}_I *\\
&\overset{(3)}{\simeq} \mathrm{map}_{\mathcal{P}(C)}(\mathrm{y}_{\mathcal{C}}(x), F)\\
&\overset{(4)}{\simeq} F(x).
\end{aligned}$$

The equivalence (1) is simply a rearrangement of the colimit operations. Note that I and J are defined by the following pullback squares

$$\begin{array}{ccc}
I & \longrightarrow & \{x\}\\
\downarrow & & \downarrow\\
J & \longrightarrow & \mathcal{C}_{x/}\\
\downarrow & & \downarrow\\
\mathcal{C}_{/F} & \longrightarrow & \mathcal{C}.
\end{array}$$

Then the equivalence (2) is a consequence of the fact that the induced monomorphism $I \to J$ is cofinal. (To see this, note that the ∞-category

$$I_{(\mathrm{y}_{\mathcal{C}}(c) \xrightarrow{g} F, x \xrightarrow{u} c)/} := J_{(\mathrm{y}_{\mathcal{C}}(c) \xrightarrow{g} F, x \xrightarrow{u} c)/} \times_J I$$

has an initial object defined by $(\mathrm{y}_{\mathcal{C}}(x) \xrightarrow{g \circ \mathrm{y}_{\mathcal{C}}(u)} F)$.) For the equivalence (3), consider the diagram of pullback squares

$$\begin{array}{ccc} I & \longrightarrow & \{x\} \\ \big\downarrow & & \big\downarrow \\ \mathcal{C}_{/F} & \longrightarrow & \mathcal{C} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{y}_{\mathcal{C}}} \\ \mathcal{P}(\mathcal{C})_{/F} & \longrightarrow & \mathcal{P}(\mathcal{C}). \end{array}$$

This shows that I is equivalent to $\mathrm{map}(\mathrm{y}_{\mathcal{C}}(x), F)$, which proves the equivalence (3). Lastly, equivalence (4) holds by the ∞-categorical Yoneda lemma (see [3, Lemma 5.1.5.2] or [2, 5.8]). □

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