∞-TOPOI – I

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1. Preliminaries

A functor $F : \mathscr{C} \to \mathscr{D}$ between ∞-categories is a **localization** if it has a fully faithful right adjoint G. Such an adjunction is determined by the endofunctor $L := GF : \mathscr{C} \to \mathscr{C}$ and the natural transformation $\alpha : 1_{\mathscr{C}} \to L$ which has the property that $L(\alpha_C), \alpha_{L(C)} : L(C) \to L^2(C)$ are equivalences. Given such a pair (L, α), $\mathscr{D}$ is equivalent to the essential image of L. See [1, 5.2.7].

A localization $F : \mathscr{C} \to \mathscr{D}$ is also determined by the class of morphisms $S_F := \{f \in \mathscr{C} \mid F(f) \text{ is an equivalence}\}$. The ∞-category $\mathscr{D}$ is equivalent to the full subcategory of $\mathscr{C}$ spanned by the S_F**-local objects**, that is, objects $X \in \mathscr{C}$ such that

$$\mathrm{map}_{\mathscr{C}}(Z, X) \xrightarrow{\simeq} \mathrm{map}_{\mathscr{C}}(Y, X)$$

for any morphism $f : Y \to Z$ in S_F. Assuming that $\mathscr{C}$ has small colimits, the class S_F is **strongly saturated**: it is closed under pushouts in $\mathscr{C}$, closed under colimits in $\mathscr{C}^{\to}$, and has the 2-out-of-3 property. See [1, 5.5.4].

Let $\mathscr{C}$ be a presentable ∞-category and S a set of morphisms. We denote $\overline{S}$ the smallest strongly saturated class of morphisms which contains S. Let $\mathscr{D}$ be the full subcategory of S-local objects. Then the inclusion $\mathscr{D} \subseteq \mathscr{C}$ has a left adjoint $F : \mathscr{C} \to \mathscr{D}$ which defines an **accessible localization**, that is, a localization such that $\mathscr{D} \subseteq \mathscr{C}$ is accessible. As a consequence, $\mathscr{D}$ is presentable. The class of morphisms which become equivalences is exactly $\overline{S}$, i.e., $S_F = \overline{S}$. Every accessible localization of $\mathscr{C}$ arises from a set of morphisms S in this way. See [1, 5.5.4].

2. Left exact localizations

A localization $F : \mathscr{C} \to \mathscr{D}$ is **left exact** if F preserves finite limits. Assuming that $\mathscr{C}$ has finite limits, a localization F is left exact if and only if the class S_F is closed under pullbacks in $\mathscr{C}$ [1, 6.2.1.1].

We may try to construct left exact accessible localizations as follows. Let $\mathscr{C}$ be a presentable ∞-category and S a set of morphisms. Let $\widetilde{S}$ denote the smallest strongly saturated class in $\mathscr{C}$ which is closed under pullbacks in $\mathscr{C}$ and contains S. Then there is an accessible localization $F : \mathscr{C} \to \mathscr{D}$ such that $S_F = \widetilde{S}$ if and only if $\widetilde{S}$ is generated by a set of morphisms as a strongly saturated class. Such a set exists if colimits in $\mathscr{C}$ are universal and pullbacks commute with filtered colimits [1, 6.2.1.2].

An ∞-category $\mathscr{X}$ is an ∞**-topos** if there exists a small ∞-category $\mathcal{C}$ and an accessible left exact localization $\mathcal{P}(\mathcal{C}) \to \mathscr{X}$. ($\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ is the ∞-category of presheaves on $\mathcal{C}$. $\mathcal{S}$ denotes the ∞-category of spaces.)

3. Descent Properties

Let $\mathscr{C}$ be a presentable ∞-category and K a simplicial set. Consider two diagrams

$$\overline{p}, \overline{q} : K^{\triangleright} \to \mathscr{C}$$

where $\overline{q}$ is a colimit diagram, and let $\overline{\alpha} : \overline{p} \to \overline{q}$ be a natural transformation whose restriction $\alpha = \overline{\alpha}_{|K}$ to K is **cartesian**, i.e., for each $x \to y$ in K, the square

$$\begin{array}{ccc} \overline{p}(x) & \longrightarrow & \overline{p}(y) \\ \downarrow & & \downarrow \\ \overline{q}(x) & \longrightarrow & \overline{q}(y) \end{array}$$

is a pullback. The ∞-category $\mathscr{C}$ satisfies **descent** if for all such quadruples $(K, \overline{p}, \overline{q}, \overline{\alpha})$, the following hold:

(D1) If $\overline{\alpha}$ is cartesian, then $\overline{p}$ is a colimit diagram.
(D2) If $\overline{p}$ is a colimit diagram, then $\overline{\alpha}$ is cartesian.

The first property (D1) essentially says that colimits in $\mathscr{C}$ are universal (see [1, Lemma 6.1.3.3]). This means that for each $f : X \to Y$, the pullback functor $f^* : \mathscr{C}_{/Y} \to \mathscr{C}_{/X}$ (which is a right adjoint) preserves colimits.

The second property (D2) says that the extension of a cartesian transformation between K-diagrams to the colimit diagrams is again cartesian – a kind of "gluing" property for cartesian transformations. Assuming (D1), property (D2) is satisfied for all K if it is satisfied for pushouts [1, 6.1.3.5–6.1.3.9].

Example 1. It is a consequence of classical results that the ∞-category $\mathcal{S}$ of spaces satisfies descent. Property (D1) can be verified in the model category of simplicial sets using the fact that it is right proper. Property (D2) follows from the classical fact that a map of spaces is a quasifibration if it is a quasifibration locally.

It follows that $\mathcal{P}(\mathcal{C})$ satisfies descent for any small ∞-category $\mathcal{C}$. The property of descent is preserved under left exact localizations, therefore every ∞-topos satisfies descent, too. See [1, 6.1.3]. An analogous study of descent in the context of model categories can be found in [2].

Example 2. Suppose that $\mathscr{C}$ satisfies (D1) and let $(1 \to X)$ be a pointed object in $\mathscr{C}$. Then the square

$$\begin{array}{ccc} \Sigma\Omega X & \longrightarrow & X \vee X \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X \end{array}$$

is a pullback square in $\mathscr{C}$.

Example 3. The nerve of $G_\bullet$ of a group defines a diagram in $\mathcal{S}$ indexed by $N(\Delta^{op})$ – this is a **groupoid object** in $\mathcal{S}$. Moreover, $G_0 \simeq \Delta^0$. We may consider the following simplicial model ('décalage') for the path space of $|G_\bullet|$, given by

$$EG_\bullet \colon N(\Delta^{op}) \to \mathcal{S}, \quad [n] \mapsto [n+1] \mapsto G_{n+1}$$

together with the 'last boundary' projection $\alpha \colon EG_\bullet \to G_\bullet$. It is easy to see that the natural transformation α is cartesian and therefore (D2) implies the well–known equivalence $G \simeq \Omega|G_\bullet|$.

Example 4. The ordinary Grothendieck (1-)topos of sets satisfies property (D1). On the other hand, it is easy to construct examples showing that (D2) is not satisfied in this category in general. Therefore the category of sets is not an ∞-topos.

4. Giraud Axioms

Let $\mathscr{C}$ be a presentable ∞-category which satisfies descent. In particular, it has the following properties:

- Colimits in $\mathscr{C}$ are universal. This is essentially a reformulation of the descent property (D1). See [1, 6.1.3.3].

  Example 5. Suppose that $\mathscr{C}$ satisfies (D1) and let $f\colon X \to 0$ be a morphism to the initial object of $\mathscr{C}$. Since the functor $f^*\colon \mathscr{C}_{/0} \to \mathscr{C}_{/X}$ preserves both limits and colimits, it follows that $f^*(0 \xrightarrow{=} 0) = (X \xrightarrow{=} X)$ is a zero object. It follows that X is an initial object in $\mathscr{C}$.

- Coproducts $\mathscr{C}$ are disjoint. This means that the squares

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \sqcup Y \end{array}$$

  are pullbacks. This is a consequence of the property in Example 5 and the descent property (D2). See [1, 6.1.3.19].
- Groupoids in $\mathscr{C}$ are effective. A **groupoid** in $\mathscr{C}$ is a simplicial object $U_\bullet : N(\Delta^{op}) \to \mathscr{C}$ such that

$$\begin{array}{ccc} U_n & \longrightarrow & U_S \\ \downarrow & & \downarrow \\ U_{S'} & \longrightarrow & U_0 \end{array}$$

  is a pullback for all decompositions $[n] = S \cup S'$ with $S \cap S' = \{s\}$ (see [1, Proposition 6.1.2.6]). A groupoid object $U_\bullet$ is called **effective** if it can be extended to a colimit diagram, given by an augmented simplicial object $U_\bullet^+ : N(\Delta_+^{op}) \to \mathscr{C}$, such that

$$\begin{array}{ccc} U_1 & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ U_0 & \longrightarrow & U_{-1} \end{array}$$

  is a pullback. Note that such an augmented simplicial object is determined by the morphism $f : U_0 \to U_{-1}$ by taking iterated pullbacks [1, Proposition 6.1.2.11]. This type of augmented simplicial object is called the **Čech nerve (of f)**. See [1, 6.1.2.7-6.1.2.15].

  The fact that groupoids in $\mathscr{C}$ are effective is a consequence of descent property (D2) – cf. Example 3. See [1, 6.1.3.19].

Conversely, these properties characterize ∞-topoi.

Theorem 6. *Let $\mathscr{X}$ be an ∞-category. The following are equivalent:*

(1) *$\mathscr{X}$ is an ∞-topos.*

(2) *$\mathscr{X}$ is a presentable ∞-category and satisfies descent.*
(3) *$\mathscr{X}$ satisfies the following:*
  (a) *$\mathscr{X}$ is a presentable ∞-category.*
  (b) *Colimits in $\mathscr{X}$ are universal.*
  (c) *Coproducts in $\mathscr{X}$ are disjoint.*
  (d) *Groupoids in $\mathscr{X}$ are effective.*

Proof. (Sketch) We sketched the proofs of (1) ⇒ (2) ⇒ (3). (3) ⇒ (1): Let $\mathscr{X}^\kappa$ be the small full subcategory of $\mathscr{X}$ spanned by the κ-compact objects. We may choose κ so that $\mathscr{X}$ is κ-presentable and κ-compact objects are closed under finite limits – in particular, $\mathscr{X}^\kappa$ contains the terminal object.

Then we obtain an accessible localization $F : \mathcal{P}(\mathscr{X}^\kappa) \to \mathscr{C}$ and it suffices to show that it is left exact. The restriction of F along the Yoneda embedding $j : \mathscr{X}^\kappa \hookrightarrow \mathcal{P}(\mathscr{X}^\kappa)$ is left exact by construction. It is a consequence of (3)(b)–(d), that the localization functor F is then again left exact [1, Proposition 6.1.5.2]. Specifically, using (3)(b), we can reduce this claim to showing that F preserves pullbacks of the form

$$\begin{array}{ccc} W & \longrightarrow & j(C) \\ \downarrow & & \downarrow \\ j(C') & \longrightarrow & Z. \end{array} \tag{1}$$

Since this holds when Z is representable (because j and $F \circ j$ are left exact), it suffices to show that the class of objects Z, for which these pullbacks are preserved, is closed under colimits. We say that Z is *good* if F preserves pullbacks as in (1) for any $C, C' \in \mathscr{X}^\kappa$ – and, as a consequence of 3(b), F preserves *every* pullback in $\mathcal{P}(\mathscr{X}^\kappa)$ whose lower right corner is Z.

(3)(c) is used to show that the class of good objects is closed under coproducts. (3)(d) is used to show that good objects are closed under coequalizers. We sketch the argument in this case: suppose that

$$U_1 \rightrightarrows U_0 \to U_{-1}$$

is a coequalizer where U_0 is good. A morphism $j(C) \to U_{-1}$ factors up to homotopy through $U_0 \to U_{-1}$. As a consequence, it is easy to see that it suffices to prove that F preserves the pullback of $(U_0 \to U_{-1} \leftarrow U_0)$.

This last assertion is shown in [1, Proposition 6.1.4.2] and the proof is based on the construction of free groupoids. In more detail, we may extend the coequalizer diagram above to an augmented simplicial object $V_\bullet$ using a left Kan extension. Then there is a **free groupoid** $W_\bullet$ generated by $V_\bullet$ together with a universal morphism of augmented simplicial objects $V_\bullet \to W_\bullet$ – the free groupoid is obtained by a process of localization, given that the property of being a groupoid can be expressed as the property of being local with respect to a set of morphisms. Using the properties of this construction, we have:

(a) $W_\bullet$ is a *groupoid resolution* (= colimit diagram associated to a groupoid object) because $V_\bullet$ is a *simplicial resolution* (= colimit diagram),
(b) the morphisms $U_{-1} \simeq V_{-1} \to W_{-1}$ and $U_0 \simeq V_0 \to W_0$ are equivalences [1, Lemma 6.1.4.6].

Therefore, the diagram

(2)
$$\begin{array}{ccc} W_1 & \longrightarrow & W_0 \\ \downarrow & & \downarrow \\ W_0 & \longrightarrow & W_{-1} \end{array}$$

is a pullback square (because groupoid objects in $\mathcal{P}(\mathscr{X}^\kappa)$ are effective), and we need to show that F sends this pullback to a pullback square in $\mathscr{X}$. Note that $F(W_\bullet)$ is a groupoid resolution (= colimit diagram associated to a groupoid object) because F preserves colimits and $U_0 \simeq W_0$ is good. Then the required result follows from the property that groupoid objects in $\mathscr{X}$ are effective. □

Remark 7. An analogous Giraud type characterization of higher topoi in the context of model categories is shown in [3, 4.9.2].

5. Classifying Objects

Suppose that $\mathscr{C}$ is a presentable ∞-category which satisfies (D1). Then (D2) admits an equivalent reformulation in terms of classifying objects as follows.

Let S be a class of morphisms in $\mathscr{C}$ which is closed under pullbacks. We say that a morphism $p\colon E \to B$ is a classifying object for S if every morphism $f\colon X \to Y$ in S is a pullback of p along a unique classifying map $c\colon Y \to B$. The question of the existence of a classifying object for S is equivalent to the question whether the following functor is representable:

$$B_S\colon \mathscr{C}^{op} \to \widehat{\mathcal{S}} = \infty\text{-category of not-necessarily small spaces}$$

$$Y \mapsto \infty\text{-groupoid of morphisms } (X \to Y) \in S.$$

Using the adjoint functor theorem for presentable ∞-categories, this happens exactly when B_S takes values in $\mathcal{S}$ and preserves small limits [1, Proposition 6.1.6.3]. The requirement that B_S preserves small limits is equivalent to (D2) for cartesian transformations $\alpha\colon p \to q$ whose components are in S.

Then the rough idea is that (D2) would hold if there is a universal morphism $p\colon E \to B$ in $\mathscr{C}$ for the class of morphisms in $\mathscr{C}$. Some obvious size restrictions are required to make the question of the existence of p meaningful:

Definition 8. We say that $f\colon X \to Y$ is relatively κ-compact if for every pullback diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

such that Y' is κ-compact, then X' is also κ-compact.

The class S_κ of relatively κ-compact morphisms is closed under pullbacks. If κ is sufficiently large, then (D2) implies that there is a classifying object for relatively κ-compact morphisms [1, Proposition 6.1.6.7]. Conversely, if these classifying object exist for all sufficiently large κ, then $\mathscr{C}$ satisfies (D2) [1, Theorem 6.1.6.8].

References

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