∞-TOPOI – II

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1. Monomorphisms

Let $\mathscr{C}$ be an ∞-category. A morphism $f : X \to Z$ is a **monomorphism** if for each object $Y \in \mathscr{C}_{/Z}$, the mapping space $\mathrm{map}_{\mathscr{C}_{/Z}}(X, Y)$ is either empty or contractible. Equivalently, for any $Y \in \mathscr{C}$, the map $\mathrm{map}_{\mathscr{C}}(Y, X) \to \mathrm{map}_{\mathscr{C}}(Y, Z)$ is up to weak equivalence an inclusion of path components. Monomorphisms are closed under pullbacks. A morphism $f : X \to Z$ is a monomorphism if and only if the associated diagonal morphism $\Delta f \colon X \xrightarrow{\simeq} X \times_Z X$ is an equivalence.

Let $\mathscr{C}$ be a presentable ∞-category and $X \in \mathscr{C}$. We denote by $\mathrm{Sub}(X)$ the class of equivalence classes of monomorphisms $U \to X$. This is a small poset which is locally presentable as an (ordinary) category [1, Proposition 6.2.1.3] – see Property (c) below.

2. Digression: Truncated Objects

Monomorphisms are a special case of a **truncated object**. Let $\mathscr{C}$ be an ∞-category and let $k \geq -1$ be an integer. We say that $X \in \mathscr{C}$ is **k-truncated** if $\mathrm{map}_{\mathscr{C}}(Y, X)$ is k-truncated for every $Y \in \mathscr{C}$, i.e., the homotopy groups of these mapping spaces vanish in degrees $> k$ (for all basepoints).

Example 1. $(X \to Z) \in \mathscr{C}_{/Z}$ is (-1)-truncated if and only if $X \to Z$ is a monomorphism in $\mathscr{C}$.

Example 2. $\mathscr{C}$ is equivalent to (the nerve of) an ordinary category if and only if every object in $\mathscr{C}$ is 0-truncated.

Let $\tau_{\leq k}\mathscr{C} \subseteq \mathscr{C}$ denote the full subcategory which is spanned by the k-truncated objects. This has the following properties [1, 5.5.6]:

(a) The full subcategory $\tau_k\mathscr{C} \subseteq \mathscr{C}$ is closed under limits. This is a consequence of the fact that k-truncated objects in the ∞-category of spaces are closed under small products and pullbacks.

(b) Let $F \colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor between ∞-categories which admit finite limits. Then F preserves k-truncated objects.

More generally, F preserves k-truncated morphisms. We say that a morphism $f \colon X \to Z$ is **k-truncated** if the homotopy fibers of the map $\mathrm{map}_{\mathscr{C}}(Y, X) \to \mathrm{map}_{\mathscr{C}}(Y, Z)$ are k-truncated for every $Y \in \mathscr{C}$. For example, $X \in \mathscr{C}$ is k-truncated if and only if the morphism $(X \to 1)$ is k-truncated.

We have the following useful observation: $f \colon X \to Z$ is k-truncated if and only if the diagonal morphism $\Delta f \colon X \to X \times_Z X$ is $(k-1)$-truncated (*Proof.* Note that it suffices to prove the claim for the ∞-category of spaces. This can be shown using standard properties of homotopy groups.) Using this fact, (b) follows by induction.

(c) Suppose that $\mathscr{C}$ is a presentable ∞-category. Then the inclusion functor $\tau_k\mathscr{C} \subseteq \mathscr{C}$ **admits a left adjoint**. This is because the k-truncated objects are the local objects with respect to the morphisms $C \otimes \partial\Delta^{k+2} \to C \otimes \Delta^{k+2}$ where C belongs to a set of objects that generate $\mathscr{C}$ under colimits. In particular, $\tau_{\leq k}\mathscr{C}$ is again presentable. We denote the localization functor by

$$\tau^{\mathscr{C}}_{\leq k} \colon \mathscr{C} \to \tau_{\leq k}\mathscr{C}.$$

(d) Let $F \colon \mathscr{C} \to \mathscr{D}$ be a functor between presentable ∞-categories. Suppose that F preserves small colimits and finite limits. Then the following diagram commutes up to canonical equivalence:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ {\scriptstyle \tau^{\mathscr{C}}_{\leq k}}\downarrow & & \downarrow{\scriptstyle \tau^{\mathscr{D}}_{\leq k}} \\ \tau_{\leq k}\mathscr{C} & \xrightarrow{F} & \tau_{\leq k}\mathscr{D}. \end{array}$$

This can be seen by noting that the associated diagram of right adjoints is well–defined and commutes (this uses (b)).

3. Effective Epimorphisms

Let $\mathscr{X}$ be an ∞-topos. A morphism $f : U \to X$ is an **effective epimorphism** if the Čech nerve $\check{C}(f)$ (= the augmented simplicial object which is defined by iterated pullbacks along the morphism f) is a **simplicial resolution** of X (= a colimit diagram).

We state some properties of effective epimorphisms in $\mathscr{X}$ [1, 6.2.3]. A morphism $f : U \to X$ is an effective epimorphism if and only if $f^* : \mathrm{Sub}(X) \to \mathrm{Sub}(U)$ is injective. The class of effective epimorphisms contains the equivalences and is closed under composition and coproducts. Furthermore, if gf is an effective epimorphism, then so is g. A morphism $f \colon U \to X$ in is an equivalence if and only if it is a monomorphism and an effective epimorphism (*Proof.* $\check{C}(f)$ is a simplicial resolution of X which is constant at U if and only if f is an effective epimorphism and a monomorphism.) In the ∞-category of spaces, a map is an effective epimorphism if and only if it is π_0-surjective.

Left exact colimit–preserving functors preserve effective monorphisms.

Proposition 3. *Let $\mathscr{X}$ be an ∞-topos, $f : V \to X$ a morphism, $V_\bullet$ the associated Čech nerve, and let $|V_\bullet|$ be the colimit of the underlying simplicial object. Then*

$$\begin{array}{ccc} V & \xrightarrow{p} & |V_\bullet| \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle j} & \\ & X & \end{array}$$

is a factorization of f into an effective epimorphism p and a monomorphism j. Moreover, j is the (–1)-truncation of f in $\mathscr{X}_{/X}$.

Proof. The morphism p is an effective epimorphism because $\mathscr{X}$ is an ∞-topos and therefore the underlying groupoid of $V_\bullet$ is effective. Consider the pullback squares

$$\begin{array}{ccc} V_n \times_{|V_\bullet|} V_m & \longrightarrow & V_n \times_X V_m \\ \downarrow & & \downarrow \\ V \times_{|V_\bullet|} V & \longrightarrow & V \times_X V \end{array}$$

The bottom morphism is an equivalence since both objects are equivalent to V_1. Hence the top morphism is an equivalence for all $m, n \geq 0$. As a consequence, $|V_\bullet| \simeq |V_\bullet| \times_{|V_\bullet|} |V_\bullet| \xrightarrow{\simeq} |V_\bullet| \times_X |V_\bullet|$ and therefore j is a monomorphism.

For every other monomorphism $j' : \widetilde{V} \to X$ that factors f, there is a homotopically unique factorization through $|V_\bullet|$ provided by $|V_\bullet| \to |\check{C}(j')| \simeq \widetilde{V}$.

See [1, Proposition 6.2.3.4]. □

4. Topological Localizations

Let $\mathscr{C}$ be a presentable ∞-category. A strongly saturated class S is called **topological** if: (a) it is generated as a strongly saturated class by monomorphisms, and (b) it is closed under pullbacks. Note that the definition does *not* require that the class is generated by a *set* of monomorphisms. But this is always true if colimits in $\mathscr{C}$ are universal as then any monomorphism in S is the colimit of monomorphisms in S whose codomains belong to a set of objects that generate $\mathscr{C}$ under colimits. See [1, Proposition 6.2.1.5].

A localization $F : \mathscr{C} \to \mathscr{D}$ is called **topological** if the class of morphisms $S_F = \{f \colon X \to Y |\ F(f) \text{ is an equivalence}\}$ is topological.

Proposition 4. *Every topological localization $F : \mathcal{P}(\mathcal{C}) \to \mathscr{D}$ is accessible and left exact.*

Proof. F is accessible because the strongly saturated class S_F is generated by a set of morphisms. It is left exact because S_F is closed under pullbacks. □

5. Grothendieck Topologies

Let $\mathcal{C}$ be an ∞-category. A **sieve on** $\mathcal{C}$ is a full subcategory $U \subseteq \mathcal{C}$ such that if $f : C \to C'$ is in $\mathcal{C}$ and $C' \in U$, then f is in U. There is bijection between the collection of sieves on $\mathcal{C}$ and the equivalence classes of (-1)-truncated object in $\mathcal{P}(\mathcal{C})$ – there are presheaves on $\mathcal{C}$ whose values are either empty or contractible. More specifically, given a (-1)-truncated presheaf F, the associated sieve is spanned by the objects $C \in \mathcal{C}$ such that $F(C) \neq \varnothing$.

A **sieve on** $C \in \mathcal{C}$ is a sieve on the ∞-category $\mathcal{C}_{/C}$. Accordingly, there is a bijection between the set of sieves on C and the set $\mathrm{Sub}(j(C))$ of equivalence classes of subobjects of the representable functor of C in $\mathcal{P}(\mathcal{C})$ – note that $\mathrm{Sub}(j(C))$ is identified with the equivalence classes of (-1)-truncated objects in $\mathcal{P}(\mathcal{C}_{/C})$ using the equivalence $\mathcal{P}(\mathcal{C}_{/C}) \simeq \mathcal{P}(\mathcal{C})_{/j(C)}$. See [1, Proposition 6.2.2.5].

A **Grothendieck topology on** $\mathcal{C}$ consists of a collection of sieves on $C \in \mathcal{C}$ for each C, called **covering sieves**, such that:

(1) $\mathcal{C}_{/C} \subseteq \mathcal{C}_{/C}$ is a covering sieve for each C.

(2) Given $f : D \to C$ and a covering sieve $U \subseteq \mathcal{C}_{/C}$, then $f^*U \subseteq \mathcal{C}_{/D}$ is a covering sieve. Here f^*U is spanned by the objects $(T \to D)$ such that $(U \to D \xrightarrow{f} C) \in U$.
(3) Given $C \in \mathcal{C}$ and a covering sieve $U \subseteq \mathcal{C}_{/C}$, a sieve $U' \subseteq \mathcal{C}_{/C}$ is a covering sieve if f^*U' is a covering sieve for each $f : D \to C$ in U.

There is a bijection between Grothendieck topologies on $\mathcal{C}$ and Grothendieck topologies on the (ordinary) homotopy category $\mathrm{h}(\mathcal{C})$ (cf. [2]). This is because the canonical functor $\mathrm{h}(\mathcal{C}_{/C}) \to \mathrm{h}(\mathcal{C})_{/C}$ is full, and therefore it induces a bijection between sieves on $C \in \mathcal{C}$ and sieves on $C \in \mathrm{h}(\mathcal{C})$.

Let $(\mathcal{C}, \tau)$ be a small ∞-category equipped with a Grothendieck topology denoted by τ. Let S_τ denote the collection of monomorphisms $U \to j(C)$ in $\mathcal{P}(\mathcal{C})$ which correspond to covering sieves. A presheaf $F \in \mathcal{P}(\mathcal{C})$ is called a **sheaf** (or **τ-sheaf**) if it is S_τ-local, i.e., for every object $C \in \mathcal{C}$ and covering sieve $U \to j(C)$, the canonical map

$$F(C) \simeq \mathrm{map}_{\mathcal{P}(\mathcal{C})}(j(C), F) \to \mathrm{map}_{\mathcal{P}(\mathcal{C})}(U, F) \simeq \lim_{(C' \to C) \in U} F(C')$$

is an equivalence. The full subcategory of τ-sheaves is denoted by $\mathrm{Sh}(\mathcal{C}, \tau)$.

Theorem 5. *Let $\mathcal{C}$ be a small ∞-category.*

(a) *Let τ be a Grothendieck topology on $\mathcal{C}$. Then $\mathrm{Sh}(\mathcal{C}, \tau)$ is a topological localization of $\mathcal{P}(\mathcal{C})$.*
(b) *There is a bijection between Grothendieck topologies on $\mathcal{C}$ and equivalence classes of topological localizations of $\mathcal{P}(\mathcal{C})$.*

Proof. (Sketch) (a) We know that the localization of $\mathcal{P}(\mathcal{C})$ at S_τ is generated by a set of monomorphisms. The proof that it is left exact is based on an explicit construction of the localization functor. The localization functor (**sheafification**) is given by a transfinite application of a functor $(-)^+ \colon F \mapsto F^+$. More specifically, for each presheaf F, the operation $(-)^+$ replaces $F(C)$, $C \in \mathcal{C}$, by the colimit over all covering sieves $U \subseteq \mathcal{C}_{/C}$ of the limit of F restricted to U:

$$F^+(C) \simeq \mathrm{colim}_{U \subseteq \mathcal{C}_{/C}} \lim_{(C' \to C) \in U} F(C').$$

See [1, 6.2.2.8–6.2.2.13]. Since the collection of covering sieves is filtered under reverse inclusion, the functor $(-)^+ \colon \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$ is left exact.

Setting $U = \mathcal{C}_{/C}$ and $(C \xrightarrow{=} C) \in U$ in the formula above, we obtain a natural transformation $\theta \colon \mathrm{id} \to (-)^+$ whose component at $F \in \mathcal{P}(\mathcal{C})$ is a morphism of simplicial presheaves $\theta_F \colon F \to F^+$. This is an S_τ-local equivalence [1, Proposition 6.2.2.14].

Iterating the operation $F \mapsto F^+$ sufficiently many times produces a τ-sheaf $L(F)$ together with an S_τ-local equivalence $F \to L(F)$. Moreover, $L \colon \mathcal{P}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}, \tau)$ is again left exact because it is a filtered colimit of left exact functors. The (transfinite) number of times the operation $(-)^+$ must be applied in order to obtain a τ-sheaf depends on the compactness ranks of the covering sieves – similarly to the standard small object argument. See [1, proof of Proposition 6.2.2.7].

(b) Given a topological localization $L : \mathcal{P}(\mathcal{C}) \to \mathscr{D}$, we say that a monomorphism $i \colon U \to j(C)$ is a covering sieve (with respect to L) if $L(i)$ is an equivalence. Since

L preserves monomorphisms, the condition that i is a covering sieve is equivalent to the condition that the morphism

$$\tau_{\leq 0} L(i) \simeq L(\tau_{\leq 0}(i))$$

is an isomorphism in $\tau_{\leq 0}\mathscr{D}$. Using that the localization is left exact, it can be shown that this collection of covering sieves defines a Grothendieck topology on $\mathcal{C}$ (or $\mathrm{h}(\mathcal{C})$). When $\mathscr{D} = \mathrm{Sh}(\mathcal{C}, \tau)$ and L is the τ-sheafification functor, this Grothendieck topology is exactly τ – since τ corresponds to the Grothendieck topology on $\mathrm{h}(\mathcal{C})$ that is associated to the left exact localization $\mathcal{P}(\mathrm{h}\mathcal{C}) \simeq \tau_{\leq 0}\mathcal{P}(\mathcal{C}) \xrightarrow{L} \tau_{\leq 0}\mathrm{Sh}(\mathcal{C}, \tau)$. See [1, 6.2.2.17]. □

The next result identifies the ∞-category of sheaves $\mathrm{Sh}(\mathcal{C}, \tau)$ in terms of a **universal property**.

Proposition 6. *Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology τ and let $L : \mathcal{P}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}, \tau)$ be the associated accessible left exact localization. Let $\mathscr{X}$ be an ∞-topos. Then the composition*

$$\mathrm{Fun}^*(\mathrm{Sh}(\mathcal{C}, \tau), \mathscr{X}) \xrightarrow{L^*} \mathrm{Fun}^*(\mathcal{P}(\mathcal{C}), \mathscr{X}) \xrightarrow{j^*} \mathrm{Fun}(\mathcal{C}, \mathscr{X})$$

is fully faithful. Here Fun^* denotes the ∞-category of left exact colimit-preserving functors.

If $\mathcal{C}$ has finite limits, then $f : \mathcal{C} \to \mathscr{X}$ is in the essential image if and only if

(a) *f is left exact, and*
(b) *for every collection $\{C_\alpha \to C\}_\alpha$ which generates a covering sieve, the morphism*

$$\coprod_\alpha f(C_\alpha) \to f(C)$$

is an effective epimorphism.

Proof. (Sketch) The functor L^* is fully faithful because L is a localization. j^* is fully faithful as a consequence of the universal property of the ∞-category of presheaves.

For the second part, suppose that the functor $f : \mathcal{C} \to \mathscr{X}$ is the restriction of $\mathcal{P}(\mathcal{C}) \xrightarrow{L} \mathrm{Sh}(\mathcal{C}, \tau) \xrightarrow{F} \mathscr{X}$. Since the Yoneda embedding is left exact, f is also left exact, so (a) is satisfied. For (b), it suffices to show that

$$\coprod_\alpha L(j(C_\alpha) \to L(j(C))$$

is an effective epimorphism – since F preserves effective epimorphisms. Consider the factorization

$$\coprod_\alpha j(C_\alpha) \xrightarrow{p} U \xrightarrow{i} j(C)$$

into an effective epimorphism p and a monomorphism i. The morphism i corresponds to the covering sieve that the collection $\{C_\alpha \to C\}$ generates. Then $L(p)$ is an effective epimorphism and $L(i)$ is an equivalence. The converse is similar. See [1, 6.2.3.20]. □

References

[1] Jacob Lurie, *Higher topos theory.* Annals of Mathematics Studies, Vol. 170. Princeton University Press, Princeton, NJ, 2009.
Online revised version: `www.math.harvard.edu/~lurie/papers/highertopoi.pdf`

[2] Bertrand Toën and Gabriele Vezzosi, *Homotopical algebraic geometry. I. Topos theory.* Adv. Math. 193 (2005), no. 2, 257–372.