∞-TOPOI – III

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1. Canonical Topologies (Is every ∞-topos an ∞-category of sheaves?)

Let $\mathscr{X}$ be an ∞-topos. We can choose a regular cardinal κ such that $\mathscr{X}$ is κ-presentable and the full subcategory of κ-compact objects $\mathscr{X}^\kappa$ is closed under finite limits. Then the inclusion functor $f\colon \mathscr{X}^\kappa \to \mathscr{X}$ extends to a left exact accessible localization $F : \mathcal{P}(\mathscr{X}^\kappa) \to \mathscr{X}$ (see [3, Proposition 6.1.5.2]).

We may consider the largest Grothendieck topology on $\mathscr{X}^\kappa$ which is compatible with this localization. More specifically, we say that a sieve $U \to j(C)$ on $C \in \mathscr{X}^\kappa$ is a covering sieve if $F(U) \to F(j(C))$ is an equivalence in $\mathscr{X}$. This defines a Grothendieck topoloy τ_f on $\mathscr{X}^\kappa$ – it is an example of a **canonical topology** which can be defined more generally for any left exact functor $f\colon \mathscr{C} \to \mathscr{X}$ (see [3, 6.2.4]). Then the functor F descends to a left exact accessible localization on the ∞-category of sheaves [3, Proposition 6.2.4.6]:

$$\begin{array}{ccc} \mathcal{P}(\mathscr{X}^\kappa) & & \\ {\scriptstyle L}\downarrow & \searrow^{F} & \\ \mathrm{Sh}(\mathscr{X}^\kappa, \tau_f) & \overset{\overline{F}}{\dashrightarrow} & \mathscr{X}. \end{array}$$

Moreover, $\overline{F}$ has the property that its right adjoint preserves effective epimorphisms.

Since $\overline{F}$ is left exact, it induces a functor on the full (reflective) subcategories of k-truncated objects, for $-1 \le k < \infty$,

$$\mathcal{P}(\mathscr{X}^\kappa)_{\le k} \rightleftarrows \mathrm{Sh}(\mathscr{X}^\kappa, \tau_f)_{\le k} \overset{\overline{F}_{\le k}}{\rightleftarrows} \mathscr{X}_{\le k}.$$

(We recall that a presheaf $X \in \mathcal{P}(\mathscr{C})$ is k-truncated if and only if its values are k-truncated spaces. See [3, 5.5.6].)

Claim 1. The induced functor $\overline{F}_{\le k}$ is an equivalence.

To prove this, it suffices to show that $\overline{F}_{\le k}$ detects equivalences. More generally, we claim that if a morphism $u : X \to Y$ in $\mathrm{Sh}(\mathscr{X}^\kappa, \tau_f)$ is l-truncated [1] and $\overline{F}(u)$ is an equivalence, then u is an equivalence. We prove this claim by induction.

For $l = -1$ (i.e., when u is a monomorphism), this holds as consequence of the density of the Yoneda embedding, the fact that colimits are universal, and the definition of the Grothendieck topology τ_f. Indeed, these ensure that we can write any monomorphism $u\colon X \to Y$ as a colimit of monomorphisms whose codomains come from representable presheaves and which are obtained as pullbacks of u.

[1] This means that the homotopy fibers of $\mathrm{map}(Z, X) \xrightarrow{u\circ -} \mathrm{map}(Z, Y)$ are l-truncated spaces for every Z in $\mathrm{Sh}(\mathscr{X}^\kappa, \tau_f)$.

For the inductive step, note first that u is l-truncated if and only if the associated diagonal morphism $\Delta(u)\colon X \to X \times_Y X$ is $(l-1)$-truncated. If $\overline{F}(u)$ is an equivalence, then $\overline{F}(\Delta(u))$ is also an equivalence because $\overline{F}$ is left exact. Hence, by induction, $\Delta(u)\colon X \to X \times_Y X$ is an equivalence, that is, u is a monomorphism. As explained above, it follows that u is an equivalence which then completes the proof of Claim 1.

The same argument shows that any left exact localization of $\mathcal{P}(\mathscr{C})_{\leq k}$ (for $k < \infty$) is topological. See [3, Proposition 6.4.1.6].

2. n-TOPOI

The ∞-category of $(n-1)$-truncated objects in an ∞-topos is an **n-topos** – this can be used as a definition. Similarly to ∞-topoi, there are several equivalent characterizations of n-topoi for $0 \leq n < \infty$ [3, Theorem 6.4.1.5]. More specifically, a presentable ∞-category $\mathscr{X}$ is an n-topos if it satisfies any of the following equivalent statements:

(1) There exists a small n-category $\mathscr{C}$ which admits finite limits, a Grothendieck topology on $\mathscr{C}$, and an equivalence of $\mathscr{X}$ with the full subcategory of $(n-1)$-truncated objects in $\mathrm{Sh}(\mathscr{C})$.
(2) There exists a small ∞-category $\mathscr{C}$ and a left exact localization
$$\mathcal{P}(\mathscr{C})_{\leq n-1} \to \mathscr{X}.$$
(3) Colimits in $\mathscr{X}$ are universal, $\mathscr{X}$ is equivalent to an n-category, and the class of $(n-2)$-truncated morphisms in $\mathscr{X}$ is local.
(4) $\mathscr{X}$ satisfies the following axioms:
   (i) $\mathscr{X}$ is equivalent to a presentable n-category.
   (ii) Colimits in $\mathscr{X}$ are universal.
   (iii) If $n > 0$, then coproducts in $\mathscr{X}$ are disjoint.
   (iv) Every n-efficient groupoid object in $\mathscr{X}$ is effective.

See [3, Theorem 6.4.1.5].

(1) ⇒ (2) is obvious. *(2) ⇒ (1)*: This follows easily from the fact that a left exact localization of $\mathcal{P}(\mathscr{C})_{\leq n-1}$ is topological (and therefore also accessible) – see the proof of Claim 1 that was sketched above.

A class S of morphisms in $\mathscr{X}$ is called **local** if it is closed under pullbacks and satisfies the descent property (D2) with respect to cartesian transformations whose components are in S. Equivalently, S is local if it is closed under coproducts and the pushout of a cartesian transformations in $\mathscr{X}^{\ulcorner}$ whose components are in S defines a cartesian transformation in $\mathscr{X}^{\square}$ whose components are in S (see [3, Definition 6.1.3.8]).

(2) ⇒ (3): Colimits in $\mathcal{P}(\mathscr{C})_{\leq n-1}$ are universal because colimits in $\mathcal{P}(\mathscr{C})$ are universal and the truncation functor $\tau_{\leq n-1}\colon \mathcal{P}(\mathscr{C}) \to \mathcal{P}(\mathscr{C})_{\leq n-1}$ preserves colimits. It follows that colimits in $\mathscr{X}$ are universal because this property is preserved under left exact localizations. Clearly, $\mathscr{X}$ is also equivalent to an n-category.

It remains to prove that the class of $(n-2)$-truncated morphisms in $\mathscr{X}$ is local. Using (2) and standard arguments based on the properties of left exact left adjoints, we may reduce to the case $\mathscr{X} = \mathcal{S}_{\leq n-1}$ (where $\mathcal{S}$ denotes the ∞-topos of spaces). Since $\mathcal{S}$ is an ∞-topos, the class of all morphisms is local. Moreover, by considering the connectivity of the fibers, it is easy to see that the class of $(n-2)$-truncated maps in $\mathcal{S}$ is also local. Then the required result follows from the key observation that applying $\tau_{\leq n-1}$ to an $(n-2)$-truncated map in $\mathcal{S}$ does not affect the fibers of the map. (This observation also explains why we cannot expect that n-topoi satisfy the same descent properties as ∞-topoi – or for that matter, m-topoi for any $m > n$.)

A groupoid object $U_\bullet$ in $\mathscr{X}$ is called ***n*-efficient** if the natural morphism $U_1 \to U_0 \times U_0$ is $(n-2)$-truncated – note that this morphism is always $(n-1)$-truncated if $\mathscr{X}$ is equivalent to an n-category.

(3) ⇒ (4): (i) and (ii) hold by assumption. The proof of (iii) is similar to one for ∞-topoi. Let $X, Y \in \mathscr{X}$. The morphism $i\colon \varnothing \to X$ is $(n-2)$-truncated if $n > 0$. Consider the pushout in $\mathscr{X}^{\ulcorner}$:

$$\begin{array}{ccc} \mathrm{id}_\varnothing & \longrightarrow & \mathrm{id}_Y \\ \downarrow & & \downarrow \\ i & \longrightarrow & j \end{array}$$

and then apply (3) to conclude that this is a cartesian transformation of $(n-2)$-truncated morphisms. Thus, the bottom arrow in the pushout is a pullback square.

(iv): Let $U_\bullet$ be an n-efficient groupoid object and let $U_\bullet^+$ be the associated augmented simplicial object given by the colimit. Let $V_\bullet^+$ be the shifted augmented simplicial object associated to $U_\bullet$ with $V_n^+ = U_{n+1}$. The augmented simplicial object $V_\bullet^+$ ('décalage') is a colimit diagram [3, Lemma 6.1.3.17]. Let cU_0 denote the constant augmented simplicial object at U_0. There is a canonical morphism of augmented simplicial objects

$$\alpha\colon V_\bullet^+ \to U_\bullet^+ \times cU_0.$$

Note that the component of α, $U_{n+1} \xrightarrow{\alpha_{[n]}} U_n \times U_0$, is a pullback of the canonical morphism $U_1 \to U_0 \times U_0$. Since $U_\bullet$ is an n-efficient groupoid, it follows that $\alpha_{|N(\Delta^{\mathrm{op}})}$ is a cartesian transformation whose components are $(n-2)$-truncated. Using (3), we conclude that the square:

$$\begin{array}{ccc} V_0 = U_1 & \longrightarrow & V_{-1} = U_0 \\ \downarrow & & \downarrow \\ U_0 \times U_0 & \longrightarrow & U_{-1}^+ \times U_0 \end{array}$$

is a pullback, which clearly implies that $U_\bullet$ is effective.

(4) ⇒ (1), (2): The proof is similar to the proof of the corresponding statement for ∞-topoi. See [3, Proposition 6.4.3.6].

Remark 2. (m-localic topoi) Let $\mathscr{X}$ be an n-topos. We have seen that there is an equivalence $\mathrm{Sh}(\mathscr{C})_{\leq n-1} \simeq \mathscr{X}$ where $\mathscr{C}$ is a small n-category with finite limits equipped with a Grothendieck topology. Using these properties of $\mathscr{C}$, it is possible to show that the restriction functor induces an equivalence of ∞-categories:

$$\mathrm{LFun}^{\mathrm{lex}}(\mathrm{Sh}(\mathscr{C})_{\leq k-1}, \mathscr{Y}) \xrightarrow{\simeq} \mathrm{LFun}^{\mathrm{lex}}(\mathrm{Sh}(\mathscr{C})_{\leq n-1}, \mathscr{Y}_{\leq n-1}) \tag{1}$$

for every k-topos $\mathscr{Y}$ and $k \geq n$. Here $\mathrm{LFun}^{\mathrm{lex}}$ denotes the ∞-category of left exact colimit–preserving functors. More generally, a k-topos (e.g., $\mathrm{Sh}(\mathscr{C})_{\leq k-1}$) is called n-**localic** when (1) is an equivalence of ∞-categories for any k-topos $\mathscr{Y}$. For example, the equivalence (1) says that every n-topos is equivalent to the subcategory of $(n-1)$-truncated objects in an n-localic k-topos for any $k \geq n$. In particular, if $\mathscr{C}$ is (the nerve of) an ordinary category with admits finite limits and is equipped with a Grothendieck topology, then the ∞-topos of sheaves $\mathrm{Sh}(\mathscr{C})$ is 1-localic. This means that for any ∞-topos $\mathscr{Y}$, we have an equivalence

$$\mathrm{LFun}^{\mathrm{lex}}(\mathrm{Sh}(\mathscr{C}), \mathscr{Y}) \simeq \mathrm{LFun}^{\mathrm{lex}}(\mathrm{Sh}(\mathscr{C})_{\leq 0}, \mathscr{Y}_{\leq 0})$$

where $\mathrm{Sh}(\mathscr{C})_{\leq 0}$ is the ordinary Grothendieck topos associated to the (ordinary) site $\mathscr{C}$. See also [3, 6.4.5].

3. Homotopy groups and ∞-connectedness

Let $\mathscr{X}$ be an ∞-topos and $X \in \mathscr{X}$. There is an object X^{S^n} together with a morphism $e : X^{S^n} \to X$ given by "evaluation" at the basepoint of S^n. The object X^{S^n} is specified by the property that there is a natural equivalence for any $Y \in \mathscr{X}$:

$$\mathrm{map}_{\mathscr{X}}(Y, X^{S^n}) \simeq \mathrm{map}(S^n, \mathrm{map}_{\mathscr{X}}(Y, X)).$$

The **n-th homotopy group** $\pi_n(X)$ of X is the 0-truncation $\tau_{\leq 0}(X^{S^n} \stackrel{e}{\to} X)$ in the ∞-topos $\mathscr{X}_{/X}$. This is a group object if $n > 0$, and an abelian group object if $n > 1$. Homotopy groups are preserved along left exact left adjoints (**geometric morphisms**) because these preserve truncated objects. For example, if $\mathscr{X} = \mathcal{S}$ and $x : * \to X$ is a basepoint, then $x^*(\pi_n(X)) \in \mathcal{S}_{\leq 0}$ is the usual n-th homotopy group $\pi_n(X, x)$. (Here x^* denotes the left adjoint of the induced geometric morphism $x^* \colon \mathcal{S}_{/X} \leftrightarrows \mathcal{S} \colon x_*$.)

More generally, we define the homotopy sets $\pi_n(f) \in (\mathscr{X}_{/X})_{\leq 0}$ of a morphism $f \colon X \to Y$ in $\mathscr{X}$ as the homotopy sets of the object $(f \colon X \to Y) \in \mathscr{X}_{/Y}$. More explicitly, we have a canonical identification:

$$\pi_n(f) \simeq \tau_{\leq 0}(X^{S^n} \times_{Y^{S^n}} Y \to X) \in (\mathscr{X}_{/X})_{\leq 0}.$$

(We have used the canonical equivalence $\mathscr{X}_{/f} \simeq \mathscr{X}_{/X}$ to view $\pi_n(f)$ as a discrete object in $\mathscr{X}_{/X}$.) These homotopy sets are again groups if $n > 0$, and abelian groups if $n > 1$. Intuitively, $\pi_n(f)$ corresponds to the n-th homotopy groups of the fibers of f. These homotopy groups have similar properties as in the classical context of topological spaces, such as, for example, the existence of long exact sequences of homotopy groups – these properties can be shown either by similar arguments internally to $\mathscr{X}$, or by reduction to the basic case $\mathcal{S}$. For $n > 0$, it is easy to check that we have a canonical isomorphism:

$$\pi_n(f) \simeq \pi_{n-1}(\Delta(f)) \tag{2}$$

where $\Delta(f) \colon X \to X \times_Y X$ is the associated diagonal morphism. See also [3, 6.5.1] and [4, Sections 8–9] for more details and further results.

If an object $X \in \mathscr{X}$ is n-truncated, then $\pi_k(X) \simeq *$ for all $k > n$. Moreover, if $n \geq 0$ and $\pi_n(X) \simeq *$, then X is $(n-1)$-truncated. More generally, if $f \colon X \to Y$ in $\mathscr{X}$ is n-truncated, then $\pi_k(f) \simeq *$ for all $k > n$. In addition, if $n \geq 0$ and $\pi_n(f) \simeq *$, then f is $(n-1)$-truncated. These claims are shown by induction using (2). Thus, the homotopy groups of f can tell whether f is k-truncated as long as we know that f is n-truncated for some arbitrarily large n. See [3, Proposition 6.5.1.7].

A morphism $f : X \to Y$ in $\mathscr{X}$ is **n-connective**, $0 \leq n \leq \infty$, if it is an effective epimorphism and $\pi_k(f) \simeq *$ for $0 \leq k < n$. A morphism $f : X \to Y$ is n-connective if and only if the associated diagonal morphism $X \to X \times_Y X$ is $(n-1)$-connective and f is an effective epimorphism – this follows from the identification in 2.
n-connective morphisms are preserved under left exact left adjoints because these are compatible with truncation functors. We also have the following important characterization of n-connective morphisms.

Proposition 3. *A morphism $f\colon X \to Y$ in $\mathscr{X}$ is n-connective if and only if its n-truncation in $\mathscr{X}_{/Y}$ is equivalent to the final object $(Y \xrightarrow{\mathrm{id}_Y} Y)$.*

Proof. See [3, Proposition 6.5.1.12] or [4, Proposition 9.8]. □

See [3, 6.5.1.12–6.5.1.20] and [4, Sections 8–9] for more results about n-connective morphisms which generalize classical results about n-connected maps in $\mathcal{S}$ to the general context of an ∞-topos.

Example 4. Let $f\colon X \to Y$ be an n-truncated ∞-connective morphism. As explained above, it follows that f is (-1)-truncated (= monomorphism). Therefore f is an equivalence.

Proposition 5. *A morphism $f\colon X \to Y$ in $\mathscr{X}$ is ∞-connective if and only if $\tau_{\leq n}(f)$ is an equivalence for every $n \geq 0$.*

Proof. See [4, Section 8 and the proof of Proposition 10.3]. It suffices to show that the canonical morphism $p_n\colon X \to \tau_{\leq n}X$ is n-connective. To see this, we verify that the object $(\tau_{\leq n}X \xrightarrow{\mathrm{id}} \tau_{\leq n}X)$ is the n-truncation of p_n in $\mathscr{X}_{/\tau_{\leq n}X}$ and then use the characterization of n-connective maps in Proposition 3. Let $(f\colon Z \to \tau_{\leq n}X)$ be an n-truncated object in $\mathscr{X}_{/\tau_{\leq n}X}$. Since $\tau_{\leq n}X$ is n-truncated in $\mathscr{X}$, it follows that Z is also n-truncated in $\mathscr{X}$. Therefore we have a diagram of mapping spaces:

$$\begin{array}{ccc} \mathrm{map}(\tau_{\leq n}X, Z) & \xrightarrow{p_n^*} & \mathrm{map}(X, Z) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{map}(\tau_{\leq n}X, \tau_{\leq n}X) & \xrightarrow{p_n^*} & \mathrm{map}(X, \tau_{\leq n}X) \end{array}$$

where the horizontal maps are equivalences. We obtain the required equivalence by passing to the vertical homotopy fibers over the point $(\tau_{\leq n}X \xrightarrow{\mathrm{id}} \tau_{\leq n}X)$. □

Proposition 6. *The class $\mathcal{W}_\infty$ of ∞-connective morphisms in $\mathscr{X}$ is a strongly saturated class generated by a set of morphisms. In addition, $\mathcal{W}_\infty$ is closed under pullbacks.*

Proof. See [3, Proposition 6.5.2.8] and [4, Proposition 10.3]. $\mathcal{W}_\infty$ is strongly saturated as a consequence of Proposition 5. The main idea for proving that $\mathcal{W}_\infty$ is of small generation is to show that for every $n \geq 0$, the class of n-connective maps is specified by accessible conditions – alternatively, the accessibility of $\mathcal{W}_\infty$ can be deduced directly from Proposition 5 since $\mathcal{W}_\infty$ can be expressed as the class of morphisms which become equivalences after applying $\tau_{\leq n}$ for every $n \geq 0$.
The last claim is a consequence of the properties of homotopy groups/truncation functors with respect to pullbacks, see [3, Proposition 6.5.1.16(6)] and [4, Proposition 8.4]. □

4. Hypercompleteness

Let $\mathscr{X}$ be an ∞-topos. An object $X \in \mathscr{X}$ is **hypercomplete** if it is local with respect to the class $\mathcal{W}_\infty$ of ∞-connective morphisms in $\mathscr{X}$. As a consequence of Proposition 6, the full subcategory of hypercomplete objects $\mathscr{X}^\wedge \subseteq \mathscr{X}$ defines a left exact accessible localization of $\mathscr{X}$ (**hypercompletion of** $\mathscr{X}$):

$$L\colon \mathscr{X} \to \mathscr{X}^\wedge.$$

In particular, $\mathscr{X}^\wedge$ is again an ∞-topos. $\mathscr{X}$ is **hypercomplete** if $\mathscr{X} = \mathscr{X}^\wedge$. The ∞-topos $\mathscr{X}^\wedge$ is hypercomplete [3, Lemma 6.5.2.12] and it is characterized by the following universal property: for every hypercomplete ∞-topos $\mathscr{Y}$, the restriction functor

$$\mathrm{LFun}^{\mathrm{lex}}(\mathscr{X}^\wedge, \mathscr{Y}) \xrightarrow{\simeq} \mathrm{LFun}^{\mathrm{lex}}(\mathscr{X}, \mathscr{Y})$$

is an equivalence between ∞-categories of left exact left adjoint functors [3, Proposition 6.5.2.13] – to see this, observe that such a functor $\mathscr{X} \to \mathscr{Y}$ preserves ∞-connective morphisms and therefore it sends ∞-connective morphisms in $\mathscr{X}$ to equivalences in $\mathscr{Y}$.

Remark 7. If $X \in \mathscr{X}$ is n-truncated, then X is hypercomplete. This follows easily from Example 4.

Example 8. The presentable ∞-category associated to the model category of simplicial presheaves with the local model structure (see [2] and [5]) is a hypercomplete ∞-topos. We recall that the weak equivalences in this model category are the morphisms of simplicial presheaves which induce isomorphisms between the sheaves of homotopy groups. See also [3, 6.5.2.14–6.5.2.15].

Proposition 9. *Let $\mathscr{X}$ be an ∞-topos, $(\mathcal{C}, \tau)$ a small ∞-category equipped with a Grothendieck topology τ, and let $F : \mathrm{Sh}(\mathcal{C}, \tau) \to \mathscr{X}$ be a left exact localization. Suppose that for every monomorphism u in $\mathrm{Sh}(\mathcal{C}, \tau)$, if $F(u)$ is an equivalence in $\mathscr{X}$, then u is an equivalence in $\mathrm{Sh}(\mathcal{C}, \tau)$. Then for every morphism u in $\mathrm{Sh}(\mathcal{C}, \tau)$, if $F(u)$ is an equivalence in $\mathscr{X}$, then u is ∞-connective in $\mathrm{Sh}(\mathcal{C}, \tau)$.*

Proof. See [3, Proposition 6.5.2.16]. Let $u : X \to Z$ be a morphism such that $F(u)$ is an equivalence. There is a factorization

$$X \xrightarrow{p} U \xrightarrow{i} Z$$

where p is an effective epimorphism and i is a monomorphism (i is the (-1)-truncation of u). It follows that $F(i)$ is an effective epimorphism and therefore an equivalence. Hence i is an equivalence by assumption. This implies that u is an effective epimorphism.

Proceeding by induction, suppose that for every morphism $u : X \to Z$ in $\mathrm{Sh}(\mathcal{C}, \tau)$, if $F(u)$ is an equivalence, then u is $(n-1)$-connective. For such a morphism u, the morphism $F(X \xrightarrow{\Delta(u)} X \times_Z X)$ is an equivalence, using that F is left exact. Then, by the inductive assumption, $\Delta(u)\colon X \to X \times_Z X$ is $(n-1)$-connective, and therefore u is n-connective. Thus, u is n-connective for every $n \geq 0$. □

Combined with the discussion in Section 1, Proposition 9 implies that every ∞-topos is obtained from an ∞-category of sheaves $\mathrm{Sh}(\mathscr{C})$ by a left exact accessible localization at a collection of ∞-connective morphisms – this is a **cotopological localization**. The maximal cotopological localization is the hypercompletion $\mathrm{Sh}(\mathscr{C})^\wedge$. See [3, 6.5.2.17–6.5.2.20].

Theorem 10. *Let $\mathscr{C}$ be a small ∞-category. There is a bijective correspondence between Grothendieck topologies on $\mathscr{C}$ and hypercomplete left exact accessible localizations of $\mathcal{P}(\mathscr{C})$. This bijection sends τ to the hypercomplete ∞-topos $\mathrm{Sh}(\mathscr{C},\tau)^\wedge$.*

Proof. See [5, Theorem 3.8.3] for the analogous statement in the context of model categories. The inverse of $\tau \mapsto \mathrm{Sh}(\mathscr{C},\tau)^\wedge$ is defined in the same way as in the case of the bijective correspondence between Grothendieck topologies on $\mathscr{C}$ and topological localizations of $\mathcal{P}(\mathscr{C})$. □

As a consequence of the classification of topological localizations of $\mathcal{P}(\mathscr{C})$ and of hypercomplete left exact accessible localizations of $\mathcal{P}(\mathscr{C})$ in terms of Grothendieck topologies on $\mathscr{C}$, it follows that if $\mathrm{Sh}(\mathscr{C},\tau)$ is not hypercomplete, then $\mathrm{Sh}(\mathscr{C},\tau)^\wedge$ is not a topological localization of $\mathcal{P}(\mathscr{C})$.

See also [4, Section 11] and [3, 6.5.4] for examples and a discussion of the difference between $\mathrm{Sh}(\mathscr{C},\tau)$ and its hypercompletion.

5. Hypercoverings

While hypercomplete objects cannot be detected in general by the covering sieves of the Grothendieck topology, an important and useful characterization of hypercomplete object in $\mathscr{X}$ is possible using the more general notion of a hypercovering. A simplicial object $U_\bullet$ in $\mathscr{X}$ is a **hypercovering** if, for every $n \geq 0$, the canonical morphism:

$$U_n \to \big(\mathrm{cosk}_{n-1}(U_\bullet)\big)_n \simeq \lim_{k\leq n-1,[k]\to[n]} U_k$$

is an effective epimorphism. For example, $U_0 \to 1$ ($n = 0$) and $U_1 \to U_0 \times U_0$ ($n = 1$) must be effective epimorphisms. The intuition is that a hypercovering $U_\bullet$ in $\mathscr{X}_{/X}$ is a generalization of the Čech nerve $V_\bullet$ of an effective epimorphism $V \to X$, in which we are allowed to replace the degree n object V_n by the domain of an effective epimorphism $U_n \to V_n$ (for example, $U_n = \coprod U_{n,i} \to V_n$ where $U_{n,i} \to V_n$ generate a covering sieve on V_n), and then continue with such refinements successively in each simplicial degree. See [1], [3, 6.5.3] and [4, 10.4].

Example 11. Let X be an ∞-connective object in $\mathscr{X}$. Then the constant simplicial object at X is a hypercovering. This is obvious when $\mathscr{X}$ is hypercomplete. The general case can be reduced to this because the hypercompletion functor $\mathscr{X} \to \mathscr{X}^\wedge$ preserves and detects effective epimorphisms (cf. Proposition 9). See [3, Lemma 6.5.3.5].

A hypercovering $U_\bullet$ in $\mathscr{X}_{/X}$ is **effective** if its colimit is a final object. For example, the Čech nerve of an effective epimorphism is an effective hypercovering. More generally,

Example 12. A simplicial object $U_\bullet$ is n-coskeletal if it is a right Kan extension of its restriction to $N(\Delta^{\mathrm{op}}_{\leq n})$. An n-coskeletal hypercovering in $\mathscr{X}_{/X}$ is effective. See [3, Lemma 6.5.3.9].

Proposition 13. *An object in $\mathscr{X}$ is hypercomplete if and only if it is local with respect to the collection S which contains the morphisms*

$$\mathrm{colim}_{N(\Delta^{\mathrm{op}})} U_\bullet \to X$$

for every X in $\mathscr{X}$ and hypercovering $U_\bullet$ in $\mathscr{X}_{/X}$.

Proof. See [1, 5]. Let $f\colon U \to X$ be an ∞-connective morphism and let $U_\bullet$ denote the constant simplicial object in $\mathscr{X}_{/X}$ with value f. Using Example 11, $U_\bullet$ is a hypercovering. Since f can be identified with the colimit of $U_\bullet$, the 'if' direction follows. For the converse, it suffices to prove that the colimit of a hypercovering $U_\bullet$ in $\mathscr{X}_{/X}$ is n-connective for every $n \geq 0$. Note that the colimit of $V_\bullet = \mathrm{cosk}_n(U_\bullet)$ is effective (by Example 12), that is, its colimit is a final object. The canonical morphism $U_\bullet \to V_\bullet$ is an equivalence in degrees $\leq n$ and it follows that the induced morphism between the colimits is n-connective (see [3, Lemma 6.5.3.10]). See also [1, 5] for more details and stronger results in this direction. $\square$

Remark 14. The last proof shows that every ∞-connective morphism $\mathscr{X}$ is equivalent to a morphism in S. Therefore, $\mathscr{X}$ is hypercomplete if every hypercovering $U_\bullet$ in $\mathscr{X}_{/X}$ is effective. The converse is also true. See [3, Theorem 6.5.3.12].

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