DIFFERENTIATION OF FUNCTORS: INTRODUCTION TO GOODWILLIE CALCULUS

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1. Recollections

Let \mathscr{C} be an ∞ -category with finite limits. The ∞ -category $\operatorname{Sp}(\mathscr{C})$ of spectrum objects in \mathscr{C} is the ∞ -category of reduced excisive functors

$$\operatorname{Sp}(\mathscr{C}) = \operatorname{Exc}_*(\mathcal{S}^{\operatorname{fin}}_*, \mathscr{C}).$$

We recall that a functor $F: \mathcal{S}_*^{\text{fin}} \to \mathscr{C}$ is **reduced** (resp. **excisive**) if F preserves the terminal object (resp. F sends pushouts to pullbacks). We have a canonical equivalence $\operatorname{Sp}(\mathscr{C}) \simeq \operatorname{Sp}(\mathscr{C}_*)$ where $\mathscr{C}_* = \mathscr{C}_{*/}$ is the associated pointed ∞ -category of pointed objects in \mathscr{C} . The equivalence holds because the values of a reduced excisive functor $F: \mathcal{S}_*^{\text{fin}} \to \mathscr{C}$ are canonically pointed.

The ∞ -category $\operatorname{Sp}(\mathscr{C})$ is stable. Note that for every $F \in \operatorname{Sp}(\mathscr{C})$, there is a canonical equivalence $F(S^0) \simeq \Omega^n F(S^n)$ for any $n \ge 0$. There is a functor $\Omega^{\infty} : \operatorname{Sp}(\mathscr{C}) \to \mathscr{C}, F \mapsto F(S^0)$. The functor Ω^{∞} is an equivalence if and only if \mathscr{C} is stable.

The stable ∞ -category Sp(\mathscr{C}) has the following **universal property of stabilization**: for any pointed ∞ -category \mathscr{D} with finite colimits, there is an equivalence

$$(\Omega^{\infty} \circ -) \colon \operatorname{Exc}_{*}(\mathscr{D}, \operatorname{Sp}(\mathscr{C})) \xrightarrow{\simeq} \operatorname{Exc}_{*}(\mathscr{D}, \mathscr{C}).$$

The construction $\mathscr{C} \mapsto \operatorname{Sp}(\mathscr{C})$ is **functorial** in the following sense: given a left exact functor $f: \mathscr{C} \to \mathscr{C}'$ between pointed ∞ -categories with finite limits, then composition with f defines a functor

$$\operatorname{Sp}(f) \colon \operatorname{Sp}(\mathscr{C}) \to \operatorname{Sp}(\mathscr{C}')$$

such that the following diagram commutes

Moreover, by the universal property of Sp(-), the functor Sp(f) is essentially unique.

Question: What can we say when $f: \mathscr{C} \to \mathscr{C}'$ is not left exact?

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2. The (First) Excisive Approximation

We make the following assumptions which apply througout this section:

- (1) \mathscr{D} is a **differentiable** ∞ -category (= \mathscr{D} has finite limits, sequential colimits, and these commute with each other, e.g., an ∞ -topos or a stable ∞ -category).
- (2) \mathscr{C} is an ∞ -category with finite colimits and a terminal object *.

Let $\operatorname{Exc}(\mathscr{C}, \mathscr{D})$ denote the ∞ -category of excisive functors from \mathscr{C} to \mathscr{D} – regarded as a full subcategory of $\operatorname{Fun}(\mathscr{C}, \mathscr{D})$.

Theorem 1. There is an adjunction

$$P_1$$
: Fun $(\mathscr{C}, \mathscr{D}) \rightleftharpoons \operatorname{Exc}(\mathscr{C}, \mathscr{D})$: incl.

Moreover, the functor P_1 is left exact.

Definition of the functor P_1 . We first consider the functor $\operatorname{Fun}(\mathscr{C},\mathscr{D}) \xrightarrow{T_1} \operatorname{Fun}(\mathscr{C},\mathscr{D})$ which is defined on objects $F \in \operatorname{Fun}(\mathscr{C},\mathscr{D})$ and $X \in \mathscr{C}$ by

$$(T_1F)(X) =$$
 pullback of $(F(*) \to F(\Sigma X) \leftarrow F(*))$.

There is a natural transformation $\theta_F \colon F \to T_1 F$ which is defined by the canonical map to the pullback. Note that θ_F is an equivalence if F is already excisive. Even though $T_1 F$ need not be excisive, the idea is that $T_1 F$ functions as a first approximate stage towards turning F into an excisive functor. The functor $T_1 \colon \operatorname{Fun}(\mathscr{C}, \mathscr{D}) \to \operatorname{Fun}(\mathscr{C}, \mathscr{D})$ is left exact because it is given by a limit of left exact functors.

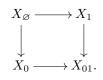
The natural transformation θ_F is also natural in F, so we obtain a natural transformation θ : id $\to T_1(-)$. Then we define:

$$P_1F = \operatorname{colim}(F \xrightarrow{\theta_F} T_1F \xrightarrow{\theta_{T_1F}} T_1(T_1F) \xrightarrow{\theta_{\dots}} \cdots).$$

(Note that we have used θ_{T_1F} instead of $T_1(\theta_F)$!) Then the resulting functor $P_1: \operatorname{Fun}(\mathscr{C}, \mathscr{D}) \to \operatorname{Fun}(\mathscr{C}, \mathscr{D})$ is again left exact because it is a sequential colimit of left exact functors and \mathscr{D} is differentiable by assumption. This proves the second claim of Theorem 1.

Proposition 2. Let $F: \mathscr{C} \to \mathscr{D}$ be a functor.

(1) Let \mathbb{X} denote a pushout diagram in \mathscr{C} :



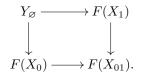
Then the morphism of squares in \mathscr{D}

$$\theta_F(\mathbb{X}) \colon F(\mathbb{X}) \to T_1F(\mathbb{X})$$

factors through a pullback square.

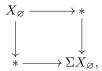
- (2) P_1F is excisive.
- (3) $P_1(\theta_F) \colon P_1(F) \xrightarrow{\simeq} P_1(T_1F).$
- (4) $P_1(F) \xrightarrow{\simeq} P_1(P_1F).$

Proof. (1) $\theta_F(\mathbb{X})$ factors through the following pullback square \mathbb{Y} :



The morphism $F(\mathbb{X}) \to \mathbb{Y}$ is the canonical morphism to the pullback. Three of the components of the morphism $\mathbb{Y} \to T_1 F(\mathbb{X})$ are defined by the natural transformation θ .

The morphism $Y_{\emptyset} \to T_1 F(X_{\emptyset})$ is defined as follows: first consider the canonical morphism of squares from X to the pushout square,



then apply F, and take pullbacks in \mathscr{D} . To see that this defines the required morphism of squares, note that each component of the morphism $\mathbb{Y} \to T_1 F(\mathbb{X})$ can also be described naturally in this way (i.e., as induced by a suitable morphism of pushouts after applying F and taking pullbacks), so we obtain a well-defined factorization of $\theta_F(\mathbb{X})$.

(2) For any pushout square X in \mathscr{C} , the square $P_1F(X)$ is a sequential colimit of morphisms of squares, and each of these morphisms factors through a pullback square by (1). As a consequence, by cofinality, $P_1F(X)$ is the sequential colimit of pullback squares in \mathscr{D} . This is again a pullback since \mathscr{D} is differentiable.

(3) We have the following composition of equivalences:

$$P_1F(-) \simeq \lim (P_1(F)(* \to \Sigma(-) \leftarrow *))$$

$$\simeq \lim (P_1(c_{F(*)} \to F(\Sigma(-)) \leftarrow c_{F(*)}))$$

$$\simeq P_1(T_1F(-))$$

where $c_{F(*)}$ denotes the constant functor at F(*) – the first equivalence uses that P_1F is excisive, the second equivalence can be verified directly, and the third equivalence uses that P_1 is left exact. (4) follows immediately from (3).

See [GoIII, Section 1], [LuHA, 6.1.1].

Proof of Theorem 1. It remains to show that $P_1: \operatorname{Fun}(\mathscr{C}, \mathscr{D}) \to \operatorname{Exc}(\mathscr{C}, \mathscr{D})$ is a localization. Let $\phi_F: F \to P_1F$ be the natural transformation induced by θ . Proposition 2(2) implies that

$$\phi_{P_1F} \colon P_1F \xrightarrow{\simeq} P_1P_1F.$$

According to Proposition 2(4), we also have that $P_1(\phi_F): P_1(F) \xrightarrow{\simeq} P_1(P_1F)$. Then the required result follows. See [GoIII, Section 1], [LuHA, 6.1.1].

 P_1F is the (1–)excisive approximation to F. Theorem 1 is the main result of [GoIII, Section 1] [LuHA, 6.1.1] in the case n = 1 – see also [GoI, Section 1].

Example. Let $F: \mathcal{S}_*^{\text{fin}} \to \mathcal{S}$ be a functor. The excisive approximation P_1F is not reduced in general. Let $P_0F: \mathcal{S}_*^{\text{fin}} \to \mathcal{S}$ denote the constant functor at F(*). There

is a natural transformation $P_1F \to P_0F$. The fiber of this natural transformation is a reduced excisive functor $D_1F \in \text{Exc}_*(\mathcal{S}^{\text{fin}}_*, \mathcal{S}_*) = \text{Sp}(\mathcal{S}_*)$ – the **differential** of F. For example, if F is the inclusion functor, then $D_1F \simeq P_1F \simeq \text{colim}_{n\to\infty}\Omega^n\Sigma^n(-)$.

3. Differentiation of Functors

Definition 3. Let $F: \mathscr{C} \to \mathscr{D}$ be a functor between ∞ -categories with finite limits. A pair $(f: \operatorname{Sp}(\mathscr{C}) \to \operatorname{Sp}(\mathscr{D}), \alpha: F \circ \Omega^{\infty}_{\mathscr{C}} \to \Omega^{\infty}_{\mathscr{D}} \circ f)$ is called a **derivative of** F if the following are satisfied:

- (a) f is exact.
- (b) For every exact functor $g: \operatorname{Sp}(\mathscr{C}) \to \operatorname{Sp}(\mathscr{D})$, the map between mapping spaces in the respective ∞ -categories of functors:

$$\operatorname{map}(f,g) \xrightarrow{(\Omega_{\mathscr{D}}^{\infty} \star -) \circ \alpha} \operatorname{map}(F \circ \Omega_{\mathscr{C}}^{\infty}, \Omega_{\mathscr{D}}^{\infty} \circ g)$$

is an equivalence.

In other words, α exhibits f as the closest approximation to a factorization of $F \circ \Omega^{\infty}_{\mathscr{G}}$ through $\Omega^{\infty}_{\mathscr{G}}$.

Example. Let $F: \mathscr{C} \to \mathscr{D}$ be a left exact functor between ∞ -categories with finite limits. Then the induced functor $\operatorname{Sp}(F): \operatorname{Sp}(\mathscr{C}) \to \operatorname{Sp}(\mathscr{D})$ together with the canonical equivalence $F \circ \Omega^{\infty}_{\mathscr{C}} \simeq \Omega^{\infty}_{\mathscr{D}} \circ \operatorname{Sp}(F)$ is the derivative of F – in this case, property (b) essentially states the universal property of $\operatorname{Sp}(-)$.

Let \mathscr{C} be an ∞ -category with finite limits and let \mathscr{D} be a differentiable ∞ -category. Let $F: \mathscr{C} \to \mathscr{D}$ be a reduced functor. The functor P_1 of Theorem 1 restricts to the ∞ -categories of reduced functors:

$$P_1: \operatorname{Fun}_*(\operatorname{Sp}(\mathscr{C}), \mathscr{D}) \to \operatorname{Exc}_*(\operatorname{Sp}(\mathscr{C}), \mathscr{D}).$$

We denote by $F' := P_1(F \circ \Omega^{\infty}_{\mathscr{C}})$ the excisive approximation to $F \circ \Omega^{\infty}_{\mathscr{C}}$ – this is also the differential of $F \circ \Omega^{\infty}_{\mathscr{C}}$. Furthermore, let $\alpha' : F \circ \Omega^{\infty}_{\mathscr{C}} \to F'$ denote the canonical natural transformation. There is an (essentially) unique exact functor $f : \operatorname{Sp}(\mathscr{C}) \to \operatorname{Sp}(\mathscr{D})$ that corresponds to F' under the equivalence:

$$\operatorname{Exc}_*(\operatorname{Sp}(\mathscr{C}), \operatorname{Sp}(\mathscr{D})) \xrightarrow{\simeq} \operatorname{Exc}_*(\operatorname{Sp}(\mathscr{C}), \mathscr{D}).$$

The natural transformation α' defines a natural transformation $\alpha \colon F \circ \Omega^{\infty}_{\mathscr{G}} \to \Omega^{\infty}_{\mathscr{G}} \circ f$. Then Theorem 1 implies:

Proposition 4. (f, α) is a derivative of F.

We denote the derivative $f: \operatorname{Sp}(\mathscr{C}) \to \operatorname{Sp}(\mathscr{D})$ of a reduced functor $F: \mathscr{C} \to \mathscr{D}$ by ∂F .

Explicit Description of the Derivative: Based on the construction of the functor P_1 , the derivative $f: \operatorname{Sp}(\mathscr{C}) \to \operatorname{Sp}(\mathscr{D})$ of the reduced functor $F: \mathscr{C} \to \mathscr{D}$ is identified with the composite functor

$$\operatorname{Sp}(\mathscr{C}) \xrightarrow{i_{\mathscr{C}} = \operatorname{incl}} \operatorname{Fun}_{\ast}(\mathcal{S}^{\operatorname{fin}}_{\ast}, \mathscr{C}) \xrightarrow{F^{+} = (F \circ -)} \operatorname{Fun}_{\ast}(\mathcal{S}^{\operatorname{fin}}_{\ast}, \mathscr{D}) \xrightarrow{L^{s}_{\mathscr{D}}} \operatorname{Sp}(\mathscr{D})$$

where $L^s_{\mathscr{D}}$ denotes the left adjoint of the inclusion functor $i_{\mathscr{D}}$ – this is just a special case of the functor P_1 . We have canonical equivalences for any $X \in \operatorname{Fun}_*(\mathcal{S}^{\operatorname{fin}}_*, \mathscr{D})$:

 $\Omega^{\infty}_{\mathscr{D}} L^{s}_{\mathscr{D}}(X) \simeq \operatorname{colim}_{n \to \infty} \Omega^{n} X(S^{n}) \text{ and } L^{s}_{\mathscr{D}}(X)(-) \simeq \operatorname{colim}_{n \to \infty} \Omega^{n} X(\Sigma^{n}(-)).$

Theorem 5 (Chain Rule). Let $\mathscr{C} \xrightarrow{F} \mathscr{D} \xrightarrow{G} \mathscr{E}$ be reduced functors between ∞ -categories with finite limits. Suppose that \mathscr{D} and \mathscr{E} are differentiable and that G preserves sequential colimits. Then there is a canonical identification:

$$\partial (G \circ F) \simeq \partial G \circ \partial F \colon \operatorname{Sp}(\mathscr{C}) \to \operatorname{Sp}(\mathscr{E})$$

Proof. See [LuHA, 6.2.1.22–6.2.1.24]. Using the description of the derivative given above, we need to show that the canonical morphism:

$$\Omega^{\infty}_{\mathscr{E}} \circ \partial(G \circ F) \simeq \Omega^{\infty}_{\mathscr{E}} L^{s}_{\mathscr{E}} G^{+} F^{+} i_{\mathscr{C}} \to \Omega^{\infty}_{\mathscr{E}} L^{s}_{\mathscr{E}} G^{+} i_{\mathscr{D}} L^{s}_{\mathscr{D}} F^{+} i_{\mathscr{C}} \simeq \Omega^{\infty}_{\mathscr{E}} \circ \partial(G) \circ \partial(F)$$

is an equivalence of functors. It suffices to prove that

(1)
$$L^s_{\mathscr{E}}G^+ \xrightarrow{\simeq} L^s_{\mathscr{E}}G^+i_{\mathscr{D}}L^s_{\mathscr{D}}.$$

In other words, we claim that for every $X \in \operatorname{Fun}_*(\mathcal{S}^{\operatorname{fin}}_*, \mathscr{D})$, the natural morphism (= unit map):

$$\eta_X \colon X \to (i_{\mathscr{D}} L^s_{\mathscr{D}})(X)$$

becomes an equivalence after applying $L^s_{\mathscr{C}}G^+$. The natural morphism (in X)

admits a natural retraction up to equivalence, induced by $G \circ \Omega_{\mathscr{D}} \to \Omega_{\mathscr{E}} \circ G$,

$$\begin{array}{ccc} \Omega^{\infty}_{\mathscr{E}}L^{s}_{\mathscr{E}}G^{+}(i_{\mathscr{D}}L^{s}_{\mathscr{D}}(X)) & \Omega^{\infty}_{\mathscr{E}}L^{s}_{\mathscr{E}}G^{+}(X) \\ \simeq & & \downarrow \\ colim_{n,m\to\infty}\Omega^{n}G(\Omega^{m}X(S^{m+n})) \longrightarrow colim_{n,m\to\infty}\Omega^{n+m}G(X(S^{m+n})) \end{array}$$

This implies that $L^s_{\mathscr{C}}G^+(\eta_X)$ is a retract up to equivalence of the morphism

$$L^s_{\mathscr{E}}G^+(i_{\mathscr{D}}L^s_{\mathscr{D}}(\eta_X)).$$

Since the last morphism is obviously an equivalence, this completes the proof that (1) is an equivalence.

References

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[GoIII] Thomas G. Goodwillie, *Calculus. III. Taylor series.* Geom. Topol. 7 (2003), 645–711. [LuHA] Jacob Lurie, *Higher Algebra.* Available online: https://www.math.ias.edu/~lurie/