DIFFERENTIATION OF FUNCTORS: INTRODUCTION TO GOODWILLIE CALCULUS

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1. Recollections

Let $\mathscr{C}$ be an ∞-category with finite limits. The **∞-category** $\mathrm{Sp}(\mathscr{C})$ **of spectrum objects in** $\mathscr{C}$ is the ∞-category of reduced excisive functors

$$\mathrm{Sp}(\mathscr{C}) = \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathscr{C}).$$

We recall that a functor $F\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathscr{C}$ is **reduced** (resp. **excisive**) if F preserves the terminal object (resp. F sends pushouts to pullbacks). We have a canonical equivalence $\mathrm{Sp}(\mathscr{C}) \simeq \mathrm{Sp}(\mathscr{C}_*)$ where $\mathscr{C}_* = \mathscr{C}_{*/}$ is the associated pointed ∞-category of pointed objects in $\mathscr{C}$. The equivalence holds because the values of a reduced excisive functor $F\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathscr{C}$ are canonically pointed.

The ∞-category $\mathrm{Sp}(\mathscr{C})$ is stable. Note that for every $F \in \mathrm{Sp}(\mathscr{C})$, there is a canonical equivalence $F(S^0) \simeq \Omega^n F(S^n)$ for any $n \geq 0$. There is a functor $\Omega^\infty\colon \mathrm{Sp}(\mathscr{C}) \to \mathscr{C}$, $F \mapsto F(S^0)$. The functor Ω^∞ is an equivalence if and only if $\mathscr{C}$ is stable.

The stable ∞-category $\mathrm{Sp}(\mathscr{C})$ has the following **universal property of stabilization**: for any pointed ∞-category $\mathscr{D}$ with finite colimits, there is an equivalence

$$(\Omega^\infty \circ -)\colon \mathrm{Exc}_*(\mathscr{D}, \mathrm{Sp}(\mathscr{C})) \xrightarrow{\simeq} \mathrm{Exc}_*(\mathscr{D}, \mathscr{C}).$$

The construction $\mathscr{C} \mapsto \mathrm{Sp}(\mathscr{C})$ is **functorial** in the following sense: given a left exact functor $f\colon \mathscr{C} \to \mathscr{C}'$ between pointed ∞-categories with finite limits, then composition with f defines a functor

$$\mathrm{Sp}(f)\colon \mathrm{Sp}(\mathscr{C}) \to \mathrm{Sp}(\mathscr{C}')$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Sp}(\mathscr{C}) & \xrightarrow{\mathrm{Sp}(f)} & \mathrm{Sp}(\mathscr{C}') \\ \downarrow{\scriptstyle \Omega^\infty_{\mathscr{C}}} & & \downarrow{\scriptstyle \Omega^\infty_{\mathscr{C}'}} \\ \mathscr{C} & \xrightarrow{f} & \mathscr{C}'. \end{array}$$

Moreover, by the universal property of $\mathrm{Sp}(-)$, the functor $\mathrm{Sp}(f)$ is essentially unique.

Question: What can we say when $f\colon \mathscr{C} \to \mathscr{C}'$ is not left exact?

2. The (First) Excisive Approximation

We make the following assumptions which apply througout this section:

1. $\mathscr{D}$ is a **differentiable** ∞-category (= $\mathscr{D}$ has finite limits, sequential colimits, and these commute with each other, e.g., an ∞-topos or a stable ∞-category).
2. $\mathscr{C}$ is an ∞-category with finite colimits and a terminal object $*$.

Let $\mathrm{Exc}(\mathscr{C}, \mathscr{D})$ denote the ∞-category of excisive functors from $\mathscr{C}$ to $\mathscr{D}$ – regarded as a full subcategory of $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$.

Theorem 1. *There is an adjunction*

$$P_1 \colon \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \rightleftarrows \mathrm{Exc}(\mathscr{C}, \mathscr{D}) \colon \mathrm{incl}.$$

Moreover, the functor P_1 is left exact.

<u>Definition of the functor P_1.</u> We first consider the functor $\mathrm{Fun}(\mathscr{C}, \mathscr{D}) \xrightarrow{T_1} \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ which is defined on objects $F \in \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ and $X \in \mathscr{C}$ by

$$(T_1 F)(X) = \text{ pullback of } (F(*) \to F(\Sigma X) \leftarrow F(*)).$$

There is a natural transformation $\theta_F \colon F \to T_1 F$ which is defined by the canonical map to the pullback. Note that θ_F is an equivalence if F is already excisive. Even though $T_1 F$ need not be excisive, the idea is that $T_1 F$ functions as a first approximate stage towards turning F into an excisive functor. The functor $T_1 \colon \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ is left exact because it is given by a limit of left exact functors.
The natural transformation θ_F is also natural in F, so we obtain a natural transformation $\theta \colon \mathrm{id} \to T_1(-)$. Then we define:

$$P_1 F = \ \mathrm{colim}(F \xrightarrow{\theta_F} T_1 F \xrightarrow{\theta_{T_1 F}} T_1(T_1 F) \xrightarrow{\theta_{\cdots}} \cdots).$$

(Note that we have used $\theta_{T_1 F}$ instead of $T_1(\theta_F)$!) Then the resulting functor $P_1 \colon \mathrm{Fun}(\mathscr{C}, \mathscr{D}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{D})$ is again left exact because it is a sequential colimit of left exact functors and $\mathscr{D}$ is differentiable by assumption. This proves the second claim of Theorem 1.

Proposition 2. *Let $F \colon \mathscr{C} \to \mathscr{D}$ be a functor.*

1. *Let $\mathbb{X}$ denote a pushout diagram in $\mathscr{C}$:*

$$\begin{array}{ccc} X_\varnothing & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & X_{01}. \end{array}$$

   Then the morphism of squares in $\mathscr{D}$

$$\theta_F(\mathbb{X}) \colon F(\mathbb{X}) \to T_1 F(\mathbb{X})$$

   factors through a pullback square.
2. *$P_1 F$ is excisive.*
3. *$P_1(\theta_F) \colon P_1(F) \xrightarrow{\simeq} P_1(T_1 F)$.*
4. *$P_1(F) \xrightarrow{\simeq} P_1(P_1 F)$.*

Proof. (1) $\theta_F(\mathbb{X})$ factors through the following pullback square $\mathbb{Y}$:

$$\begin{array}{ccc} Y_\varnothing & \longrightarrow & F(X_1) \\ \downarrow & & \downarrow \\ F(X_0) & \longrightarrow & F(X_{01}). \end{array}$$

The morphism $F(\mathbb{X}) \to \mathbb{Y}$ is the canonical morphism to the pullback. Three of the components of the morphism $\mathbb{Y} \to T_1F(\mathbb{X})$ are defined by the natural transformation θ.

The morphism $Y_\varnothing \to T_1F(X_\varnothing)$ is defined as follows: first consider the canonical morphism of squares from $\mathbb{X}$ to the pushout square,

$$\begin{array}{ccc} X_\varnothing & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X_\varnothing, \end{array}$$

then apply F, and take pullbacks in $\mathscr{D}$. To see that this defines the required morphism of squares, note that each component of the morphism $\mathbb{Y} \to T_1F(\mathbb{X})$ can also be described naturally in this way (i.e., as induced by a suitable morphism of pushouts after applying F and taking pullbacks), so we obtain a well–defined factorization of $\theta_F(\mathbb{X})$.

(2) For any pushout square $\mathbb{X}$ in $\mathscr{C}$, the square $P_1F(\mathbb{X})$ is a sequential colimit of morphisms of squares, and each of these morphisms factors through a pullback square by (1). As a consequence, by cofinality, $P_1F(\mathbb{X})$ is the sequential colimit of pullback squares in $\mathscr{D}$. This is again a pullback since $\mathscr{D}$ is differentiable.

(3) We have the following composition of equivalences:

$$\begin{aligned} P_1F(-) &\simeq \lim\big(P_1(F)\big(* \to \Sigma(-) \leftarrow *\big)\big) \\ &\simeq \lim\big(P_1\big(c_{F(*)} \to F(\Sigma(-)) \leftarrow c_{F(*)}\big)\big) \\ &\simeq P_1\big(T_1F(-)\big) \end{aligned}$$

where $c_{F(*)}$ denotes the constant functor at $F(*)$ – the first equivalence uses that P_1F is excisive, the second equivalence can be verified directly, and the third equivalence uses that P_1 is left exact. (4) follows immediately from (3).

See [GoIII, Section 1], [LuHA, 6.1.1]. □

Proof of Theorem 1. It remains to show that $P_1\colon \mathrm{Fun}(\mathscr{C},\mathscr{D}) \to \mathrm{Exc}(\mathscr{C},\mathscr{D})$ is a localization. Let $\phi_F\colon F \to P_1F$ be the natural transformation induced by θ. Proposition 2(2) implies that

$$\phi_{P_1F}\colon P_1F \xrightarrow{\simeq} P_1P_1F.$$

According to Proposition 2(4), we also have that $P_1(\phi_F)\colon P_1(F) \xrightarrow{\simeq} P_1(P_1F)$. Then the required result follows. See [GoIII, Section 1], [LuHA, 6.1.1]. □

P_1F is the **(1–)excisive approximation** to F. Theorem 1 is the main result of [GoIII, Section 1] [LuHA, 6.1.1] in the case $n = 1$ – see also [GoI, Section 1].

Example. Let $F\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ be a functor. The excisive approximation P_1F is not reduced in general. Let $P_0F\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ denote the constant functor at $F(*)$. There is a natural transformation $P_1F \to P_0F$. The fiber of this natural transformation is a reduced excisive functor $D_1F \in \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}_*) = \mathrm{Sp}(\mathcal{S}_*)$ – the **differential** of F. For example, if F is the inclusion functor, then $D_1F \simeq P_1F \simeq \mathrm{colim}_{n\to\infty}\Omega^n\Sigma^n(-)$.

3. Differentiation of Functors

Definition 3. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor between ∞-categories with finite limits. A pair $(f\colon \mathrm{Sp}(\mathscr{C}) \to \mathrm{Sp}(\mathscr{D}), \alpha\colon F \circ \Omega^\infty_{\mathscr{C}} \to \Omega^\infty_{\mathscr{D}} \circ f)$ is called a **derivative of** F if the following are satisfied:

(a) f is exact.
(b) For every exact functor $g\colon \mathrm{Sp}(\mathscr{C}) \to \mathrm{Sp}(\mathscr{D})$, the map between mapping spaces in the respective ∞-categories of functors:
$$\mathrm{map}(f, g) \xrightarrow{(\Omega^\infty_{\mathscr{D}} \star -)\circ\alpha} \mathrm{map}(F \circ \Omega^\infty_{\mathscr{C}}, \Omega^\infty_{\mathscr{D}} \circ g)$$
is an equivalence.

In other words, α exhibits f as the closest approximation to a factorization of $F \circ \Omega^\infty_{\mathscr{C}}$ through $\Omega^\infty_{\mathscr{D}}$.

Example. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a left exact functor between ∞-categories with finite limits. Then the induced functor $\mathrm{Sp}(F)\colon \mathrm{Sp}(\mathscr{C}) \to \mathrm{Sp}(\mathscr{D})$ together with the canonical equivalence $F \circ \Omega^\infty_{\mathscr{C}} \simeq \Omega^\infty_{\mathscr{D}} \circ \mathrm{Sp}(F)$ is the derivative of F – in this case, property (b) essentially states the universal property of $\mathrm{Sp}(-)$.

Let $\mathscr{C}$ be an ∞-category with finite limits and let $\mathscr{D}$ be a differentiable ∞-category. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a reduced functor. The functor P_1 of Theorem 1 restricts to the ∞-categories of reduced functors:
$$P_1\colon \mathrm{Fun}_*(\mathrm{Sp}(\mathscr{C}), \mathscr{D}) \to \mathrm{Exc}_*(\mathrm{Sp}(\mathscr{C}), \mathscr{D}).$$
We denote by $F'\colon = P_1(F \circ \Omega^\infty_{\mathscr{C}})$ the excisive approximation to $F \circ \Omega^\infty_{\mathscr{C}}$ – this is also the differential of $F \circ \Omega^\infty_{\mathscr{C}}$. Furthermore, let $\alpha'\colon F \circ \Omega^\infty_{\mathscr{C}} \to F'$ denote the canonical natural transformation. There is an (essentially) unique exact functor $f\colon \mathrm{Sp}(\mathscr{C}) \to \mathrm{Sp}(\mathscr{D})$ that corresponds to F' under the equivalence:
$$\mathrm{Exc}_*(\mathrm{Sp}(\mathscr{C}), \mathrm{Sp}(\mathscr{D})) \xrightarrow{\simeq} \mathrm{Exc}_*(\mathrm{Sp}(\mathscr{C}), \mathscr{D}).$$
The natural transformation α' defines a natural transformation $\alpha\colon F \circ \Omega^\infty_{\mathscr{C}} \to \Omega^\infty_{\mathscr{D}} \circ f$. Then Theorem 1 implies:

Proposition 4. *(f, α) is a derivative of F.*

We denote the derivative $f\colon \mathrm{Sp}(\mathscr{C}) \to \mathrm{Sp}(\mathscr{D})$ of a reduced functor $F\colon \mathscr{C} \to \mathscr{D}$ by ∂F.

Explicit Description of the Derivative: Based on the construction of the functor P_1, the derivative $f\colon \mathrm{Sp}(\mathscr{C}) \to \mathrm{Sp}(\mathscr{D})$ of the reduced functor $F\colon \mathscr{C} \to \mathscr{D}$ is identified with the composite functor
$$\mathrm{Sp}(\mathscr{C}) \xrightarrow{i_{\mathscr{C}} = \mathrm{incl}} \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathscr{C}) \xrightarrow{F^+ = (F\circ -)} \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathscr{D}) \xrightarrow{L^s_{\mathscr{D}}} \mathrm{Sp}(\mathscr{D})$$
where $L^s_{\mathscr{D}}$ denotes the left adjoint of the inclusion functor $i_{\mathscr{D}}$ – this is just a special case of the functor P_1. We have canonical equivalences for any $X \in \mathrm{Fun}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathscr{D})$:
$$\Omega^\infty_{\mathscr{D}} L^s_{\mathscr{D}}(X) \simeq \mathrm{colim}_{n\to\infty}\Omega^n X(S^n) \text{ and } L^s_{\mathscr{D}}(X)(-) \simeq \mathrm{colim}_{n\to\infty}\Omega^n X(\Sigma^n(-)).$$

Theorem 5 (Chain Rule)**.** *Let $\mathscr{C} \xrightarrow{F} \mathscr{D} \xrightarrow{G} \mathscr{E}$ be reduced functors between ∞-categories with finite limits. Suppose that $\mathscr{D}$ and $\mathscr{E}$ are differentiable and that G preserves sequential colimits. Then there is a canonical identification:*

$$\partial(G \circ F) \simeq \partial G \circ \partial F \colon \mathrm{Sp}(\mathscr{C}) \to \mathrm{Sp}(\mathscr{E}).$$

Proof. See [LuHA, 6.2.1.22–6.2.1.24]. Using the description of the derivative given above, we need to show that the canonical morphism:

$$\Omega^\infty_{\mathscr{E}} \circ \partial(G \circ F) \simeq \Omega^\infty_{\mathscr{E}} L^s_{\mathscr{E}} G^+ F^+ i_{\mathscr{C}} \to \Omega^\infty_{\mathscr{E}} L^s_{\mathscr{E}} G^+ i_{\mathscr{D}} L^s_{\mathscr{D}} F^+ i_{\mathscr{C}} \simeq \Omega^\infty_{\mathscr{E}} \circ \partial(G) \circ \partial(F)$$

is an equivalence of functors. It suffices to prove that

$$L^s_{\mathscr{E}} G^+ \xrightarrow{\simeq} L^s_{\mathscr{E}} G^+ i_{\mathscr{D}} L^s_{\mathscr{D}}. \tag{1}$$

In other words, we claim that for every $X \in \mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathscr{D})$, the natural morphism (= unit map):

$$\eta_X \colon X \to (i_{\mathscr{D}} L^s_{\mathscr{D}})(X)$$

becomes an equivalence after applying $L^s_{\mathscr{E}} G^+$. The natural morphism (in X)

$$\begin{array}{ccc} \Omega^\infty_{\mathscr{E}} L^s_{\mathscr{E}} G^+(X) & \longrightarrow & \Omega^\infty_{\mathscr{E}} L^s_{\mathscr{E}} G^+(i_{\mathscr{D}} L^s_{\mathscr{D}}(X)) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{colim}_{n\to\infty} \Omega^n G(X(S^n)) & \longrightarrow & \mathrm{colim}_{n,m\to\infty} \Omega^n G(\Omega^m X(S^{m+n})) \end{array}$$

admits a natural retraction up to equivalence, induced by $G \circ \Omega_{\mathscr{D}} \to \Omega_{\mathscr{E}} \circ G$,

$$\begin{array}{ccc} \Omega^\infty_{\mathscr{E}} L^s_{\mathscr{E}} G^+(i_{\mathscr{D}} L^s_{\mathscr{D}}(X)) & & \Omega^\infty_{\mathscr{E}} L^s_{\mathscr{E}} G^+(X) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{colim}_{n,m\to\infty} \Omega^n G(\Omega^m X(S^{m+n})) & \longrightarrow & \mathrm{colim}_{n,m\to\infty} \Omega^{n+m} G(X(S^{m+n})). \end{array}$$

This implies that $L^s_{\mathscr{E}} G^+(\eta_X)$ is a retract up to equivalence of the morphism

$$L^s_{\mathscr{E}} G^+(i_{\mathscr{D}} L^s_{\mathscr{D}}(\eta_X)).$$

Since the last morphism is obviously an equivalence, this completes the proof that (1) is an equivalence. □

References

[GoI] Thomas G. Goodwillie, *Calculus. I. The first derivative of pseudoisotopy theory.* K-Theory 4 (1990), no. 1, 1–27.

[GoIII] Thomas G. Goodwillie, *Calculus. III. Taylor series.* Geom. Topol. 7 (2003), 645–711.

[LuHA] Jacob Lurie, *Higher Algebra.* Available online: `https://www.math.ias.edu/~lurie/`