PRESENTABLE ∞-CATEGORIES – I

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1. Preliminaries

1.1. **Adjoint functors.** Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories. G admits a **left adjoint** if and only if for any $c \in \mathcal{C}$ the ∞-category $\mathcal{D}_{c/}$ defined by the pullback:

$$\begin{array}{ccc} \mathcal{D}_{c/} & \longrightarrow & \mathcal{C}_{c/} \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow[G]{} & \mathcal{C} \end{array}$$

admits an initial object. A proof of this can be found near [1, Proposition 5.2.4.2].

An ∞-category $\mathcal{C}$ admits an **initial object** if the following conditions are satisfied:

(1) $\mathcal{C}$ is locally small.
(2) $\mathcal{C}$ is complete.
(3) $\mathcal{C}$ admits a **small weakly initial set**, i.e., there is a small set of objects S in $\mathcal{C}$ with the property that for every $x \in \mathcal{C}$ there exists $s \in S$ such that $\mathrm{map}(s, x) \neq \emptyset$.

A proof of this can be found in [2, Proposition 2.3.2].

1.2. **Localizations.** A **localization** is a functor $F : \mathcal{C} \to \mathcal{D}$ which admits a fully faithful right adjoint (equivalently: the counit of the adjunction is an equivalence). Given an ∞-category $\mathcal{C}$ and a functor $L : \mathcal{C} \to \mathcal{C}$, the following are equivalent:

(1) The functor $\mathcal{C} \xrightarrow{L} L\mathcal{C}$ is a localization, that is, it is a left adjoint of the inclusion functor $L\mathcal{C} \subseteq \mathcal{C}$. Here $L\mathcal{C}$ denotes the full subcategory spanned by the essential image of L.
(2) There is a natural transformation $\alpha : \mathrm{id}_{\mathcal{C}} \to L$ such that for every $C \in \mathcal{C}$, the canonical morphisms $\alpha_{L(C)}$ and $L(\alpha_C)$ are equivalences.

In this case, we will often call the functor L also a localization. Moreover, we have: $C \in L\mathcal{C} \Leftrightarrow C$ is L-local (= local with respect to the class of maps f in $\mathcal{C}$ such that $L(f)$ is an equivalence). See [1, 5.2.7].

2. Characterization of Presentable ∞-Categories

Theorem 1. *Let $\mathcal{C}$ be an ∞-category. Then the following statements are equivalent:*

(1) $\mathcal{C}$ is accessible and cocomplete.
(2) $\mathcal{C}$ is accessible and for every regular cardinal κ the full subcategory of κ-compact objects $\mathcal{C}^\kappa$ admits κ-small colimits.
(3) There exists a regular cardinal κ such that $\mathcal{C}$ is κ-accessible and $\mathcal{C}^\kappa$ admits κ-small colimits.

(4) *For some regular cardinal κ, there exists a small ∞-category $\mathcal{D}$ which admits κ-small colimits and $\mathrm{Ind}_\kappa(\mathcal{D}) \simeq \mathcal{C}$.*
(5) *There exists a small ∞-category $\mathcal{D}$ such that $\mathcal{C}$ is an accessible localization of $\mathcal{P}(\mathcal{D})$.*
(6) *$\mathcal{C}$ is locally small, cocomplete, and for some regular cardinal κ, there exists a set S of κ-compact objects in $\mathcal{C}$ such that S generates $\mathcal{C}$ under small colimits.*
(7) *$\mathcal{C}$ is locally small, cocomplete, and for some regular cardinal κ, there exists a set S of κ-compact objects in $\mathcal{C}$ which jointly detect equivalences (= a morphism $u : c \to c'$ in $\mathcal{C}$ is an equivalence if and only if $\mathrm{map}(s, u)$ is an equivalence for every $s \in S$).*

If these equivalent conditions are satisfied, we call $\mathcal{C}$ ***presentable****.*

Lemma 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be accessible ∞-categories and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction. Then F and G are accessible functors.*

Proof. F is accessible since left adjoints preserve colimits. We may assume that $\mathcal{C}$ and $\mathcal{D}$ are both κ-accessible. As $\mathcal{C}^\kappa$ is essentially small, there is a $\tau >> \kappa$ such that $F(\mathcal{C}^\kappa) \subseteq \mathcal{D}^\tau$. Let $p : I \to \mathcal{D}, i \mapsto d_i$ be a τ-filtered diagram in $\mathcal{D}$ with colimit $d \in \mathcal{D}$. For each $c \in \mathcal{C}$, there is a (κ-filtered) diagram $J \to \mathcal{C}^\kappa, j \mapsto c_j$, with colimit c. Then:

$$\begin{aligned}\mathrm{map}(c, \mathrm{colim}_I(G \circ p)) &\simeq \lim_{j\in J} \mathrm{map}(c_j, \mathrm{colim}_{i\in I}(G(d_i)))\\ &\simeq \lim_{j\in J} \mathrm{colim}_{i\in I}\, \mathrm{map}(c_j, G(d_i))\\ &\simeq \lim_{j\in J} \mathrm{colim}_{i\in I}\, \mathrm{map}(F(c_j), d_i)\\ &\simeq \lim_{j\in J} \mathrm{map}(F(c_j), d)\\ &\simeq \mathrm{map}(F(c), d) \simeq \mathrm{map}(c, G(d))\end{aligned}$$

This implies that G preserves τ-filtered colimits. □

Proposition 3. *Let $\mathcal{C}$ be an accessible ∞-category and let $L : \mathcal{C} \to \mathcal{C}$ be a localization. The following are equivalent:*

(1) *L is accessible.*
(2) *$L\mathcal{C}$ is accessible.*

Proof. *(1) ⇒ (2)*: Assume that L is κ-continuous (= preserves κ-filtered colimits) and $\mathcal{C}$ is κ-accessible. There is $\tau >> \kappa$ such that $L(\mathcal{C}^\kappa) \subseteq (L\mathcal{C})^\tau$, as $\mathcal{C}^\kappa$ is essentially small. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ which is spanned by colimits of τ-small κ-filtered diagrams with values in $\mathcal{C}^\kappa$. Then $L(\mathcal{C}')$ still consists of τ-compact objects in $L\mathcal{C}$, as L preserves κ-filtered colimits and $\mathcal{C}^\tau$ is closed under τ-small colimits. Every object in $\mathcal{C}$ is a κ-filtered colimit of objects in $\mathcal{C}^\kappa$. By adding vertices (= cones) to this κ-filtered index category so that it becomes τ-filtered, and extending the diagram by taking τ-small colimits, we can also write an object of $\mathcal{C}$ as a τ-filtered colimit of objects in $\mathcal{C}'$ (see [1, Proposition 4.2.3.4 and the proof of Proposition 5.4.2.9]). As $L : \mathcal{C} \to L\mathcal{C}$ preserves colimits, it follows that $L(\mathcal{C}')$ generates $L\mathcal{C}$ under τ-filtered colimits. *(2) ⇒ (1)*: L is the composition of adjoint functors between accessible ∞-categories, so the claim follows from the preceding lemma.
See [1, Proposition 5.5.1.2]. □

Digression 4 (Idempotent completeness)**.** Let Idem^+ be the nerve of the ordinary category with two objects X and Y and morphisms

$$\mathrm{Hom}(X,X) = \{\mathrm{id}_X, e\} \qquad \mathrm{Hom}(Y,Y) = \{\mathrm{id}_Y\}$$
$$\mathrm{Hom}(X,Y) = \{r\} \qquad \mathrm{Hom}(Y,X) = \{s\}$$

such that $s \circ r = e$ and $e \circ e = e$. Let $\mathrm{Idem} \subseteq \mathrm{Idem}^+$ be the full subcategory spanned by X. We call an ∞-category $\mathcal{C}$ **idempotent complete** if the restriction functor

$$\mathrm{Fun}(\mathrm{Idem}^+, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Idem}, \mathcal{C})$$

is a trivial fibration. See [1, 4.4.5].

We will use the following facts:

(i) For a small ∞-category $\mathcal{C}$, there is a fully faithful functor $F\colon \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is small and idempotent complete, and every object $D \in \mathcal{D}$ is a retract of $F(C)$ for some $C \in \mathcal{C}$. $\mathcal{D}$ is called the **idempotent completion of** $\mathcal{C}$. See [1, 5.1.4].
The functor induced by F between the Ind_κ-completions is an equivalence of ∞-categories [1, Proposition 5.5.1.3].

(ii) An idempotent complete subcategory of an idempotent complete ∞-category is closed under retracts.

Proof. (of Theorem 1): (See [1, Theorem 5.5.1.1])

(1)⇒(2) follows from the fact that κ-compact objects are closed under κ-small colimits (which exist in $\mathcal{C}$) [1, Corollary 5.3.4.15]. *(2)⇒(3)* is obvious. *(3)⇒(4)* follows from the equivalence $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \simeq \mathcal{C}$.

(4)⇒(5): By Digression 4, we may assume that $\mathcal{D}$ is idempotent complete.

Claim: The Yoneda embedding $j : \mathcal{D} \to \mathcal{P}^\kappa(\mathcal{D})$ is a right adjoint.

Proof of Claim: According to the results of Subsection 1.1, it suffices to show that for every $F \in \mathcal{P}^\kappa(\mathcal{D})$, the functor $\mathrm{map}(F, \_) \circ j\colon \mathcal{D} \to \mathcal{S}$ is corepresentable. We know that F is a retract of a colimit of a κ-small diagram of representable functors $I \to \mathcal{P}(\mathcal{D}), i \mapsto j(d_i)$, see [1, Proposition 5.3.4.17]. Thus, $\mathrm{map}(F, \_) \circ j$ is a retract of the corepresentable functor

$$\lim_{i \in I} \mathrm{map}(j(d_i), j(\_)) \simeq \lim_{i \in I} \mathrm{map}(d_i, \_) \simeq \mathrm{map}(\mathrm{colim}_{i \in I}\, d_i, \_).$$

As $\mathcal{D}$ is idempotent complete, it follows that a retract of a corepresentable functor is again corepresentable. This completes the proof of the Claim.

Note that j is also fully faithful. Let $\mathcal{D}'$ be a small ∞-category which is equivalent to $\mathcal{P}^\kappa(\mathcal{D})$. By applying Ind_κ to the adjunction of the Claim, we obtain a new adjunction with a fully faithful right adjoint and a left adjoint:

$$L\colon \mathcal{P}(\mathcal{D}) \simeq \mathrm{Ind}_\kappa(\mathcal{D}') \rightleftarrows \mathrm{Ind}_\kappa(\mathcal{D}) \simeq \mathcal{C}.$$

(5)⇒(6): Let F and G denote the left and right adjoints of the adjunction in (5). Note that $\mathcal{C}$ is locally small since $\mathcal{P}(\mathcal{D})$ is locally small. We claim that every small diagram $p : I \to \mathcal{C}$ has a colimit. We know that $G \circ p$ has a colimit in $\mathcal{P}(\mathcal{D})$. As F preserves colimits, the composite diagram $FG \circ p$ has a colimit in $\mathcal{C}$. This diagram is equivalent to p, as FG is equivalent to $\mathrm{id}_\mathcal{C}$ (witnessed by the counit transformation). This shows that $\mathcal{C}$ is cocomplete.

Let $S := \{F(j(d)) \mid d \in \mathcal{D}\}$. Choose a regular cardinal κ such that G is κ-continuous. Then for all $d \in \mathcal{D}$, $F(j(d)) \in \mathcal{C}$ is κ-compact, because the functor

$$\operatorname{map}(F(j(d)), \_) \simeq \operatorname{map}(j(d), G(\_))$$

commutes with κ-filtered colimits. Moreover, since $\{j(d) \mid d \in \mathcal{D}\}$ generates $\mathcal{P}(\mathcal{D})$ under small colimits, it follows that S also generates $\mathcal{C}$.

(6)⇒(1): Let $\mathcal{C}' \subseteq \mathcal{C}$ denote the full subcategory of κ-small colimits of diagrams in S. Then $\mathcal{C}'$ consists of κ-compact objects, as those are closed under κ-small colimits. We can rewrite every small colimit of a diagram $p\colon I \to \mathcal{C}$ with values in S as the colimit of a κ-filtered diagram $p'\colon I' \to \mathcal{C}'$ – we may choose I' to be the poset of κ-small subsets of I and define p' by taking colimits. Thus, $\mathcal{C}$ is accessible.

See also [1, Theorem 5.5.1.1].

(6)⇒(7): Let S be a set of κ-compact objects as in (6). We claim that S detects equivalences. Suppose that $u : a \to b$ is a morphism in $\mathcal{C}$ such that $\operatorname{map}(s, u)$ is an equivalence for every $s \in S$. The class of objects $c \in \mathcal{C}$ for which $\operatorname{map}(c, u)$ is an equivalence is closed under colimits in $\mathcal{C}$. It follows that this class contains all the objects of $\mathcal{C}$, which then implies that u is an equivalence.

(7)⇒(6) – First Proof. (G. Raptis) Let $\mathcal{D}$ be the smallest full subcategory of $\mathcal{C}$ which contains S and is closed under κ-small colimits. Let $c \in \mathcal{C}$ and consider the associated canonical κ-filtered diagram with respect to $\mathcal{D}$:

$$J_c\colon \mathcal{D}_{/c} \to \mathcal{D} \to \mathcal{C},\ (d \to c) \mapsto d.$$

We claim that the canonical morphism $u\colon \operatorname{colim}_{\mathcal{D}_{/c}} J_c \to c$ is an equivalence. By assumption, u an equivalence if $\operatorname{map}_{\mathcal{C}}(s, u)$ is an equivalence for every $s \in S$. By adjunction, it suffices to prove that u has the right lifting property with respect to the morphisms

$$s \otimes \partial\Delta^n \to s \otimes \Delta^n$$

for every $s \in S$ and $n \geq 0$. Note that these tensors exist because $\mathcal{C}$ is cocomplete and the morphisms are in $\mathcal{D}$. Consider a lifting problem in $\mathcal{C}$ of the following form:

$$\begin{array}{ccc} s \otimes \partial\Delta^n & \longrightarrow & \operatorname{colim}_{\mathcal{D}_{/c}} J_c \\ \downarrow & & \downarrow{\scriptstyle u} \\ s \otimes \Delta^n & \longrightarrow & c. \end{array}$$

Since $s \otimes \partial\Delta^n$ is again κ-compact and $\mathcal{D}_{/c}$ is κ-filtered, the top arrow factors through some stage $J_c(d \to c)$. We may then consider the following pushout in $\mathcal{C}$

$$\begin{array}{ccccc} s \otimes \partial\Delta^n & \longrightarrow & d & & \\ \downarrow & & \downarrow & \searrow & \\ s \otimes \Delta^n & \dashrightarrow & d' & \dashrightarrow & c. \\ & \searrow & & \nearrow & \end{array}$$

The morphism $(d' \to c)$ defines an object in $\mathcal{D}_{/c}$. So, by definition, this morphism factors canonically through u giving the required lift in our original lifting problem.

Second Proof. (D.–C. Cisinski) Let $\mathcal{C}^\kappa \subseteq \mathcal{C}$ be the full subcategory of κ-compact objects. Then $\mathcal{C}^\kappa$ contains S and is closed under κ-small colimits. Let $c \in \mathcal{C}$ and consider the associated canonical κ-filtered diagram with respect to $\mathcal{C}^\kappa$:

$$J_c \colon \mathcal{C}^\kappa_{/c} \to \mathcal{C}^\kappa \to \mathcal{C}.$$

We claim that the canonical morphism $\operatorname{colim}_{\mathcal{C}^\kappa_{/c}} J_c \to c$ is an equivalence. For every $s \in S$, we have a canonical equivalence:

$$\operatorname{map}(s, \operatorname{colim}_{\mathcal{C}^\kappa_{/c}} J_c) \simeq \operatorname{colim}_{\mathcal{C}^\kappa_{/c}} \operatorname{map}(s, J_c(\_))$$

because s is κ-compact. Moreover, there is a canonical equivalence:

$$\operatorname{colim}_{(c' \to c) \in \mathcal{C}^\kappa_{/c}} \operatorname{map}(\_, c') \simeq \operatorname{map}(\_, c)$$

using the density of the Yoneda embedding for $\mathcal{C}^\kappa$ and the Yoneda Lemma for $\mathcal{C}$. Since S detects equivalences, the required result follows. □

Corollary 5. *Let $\mathcal{C}$ be a presentable ∞-category. Then $\mathcal{C}$ is complete.*

Proof. By Theorem 1(5), we may assume that $\mathcal{C}$ is the full subcategory of local objects in $\mathcal{P}(\mathcal{D})$ with respect to a localization $L \colon \mathcal{P}(\mathcal{D}) \rightleftarrows \mathcal{C} \colon i$. Let $p : I \to \mathcal{C}$ be a small diagram. Local objects are closed under limits in $\mathcal{P}(\mathcal{D})$, hence the limit of the diagram $i \circ p$ in $\mathcal{P}(\mathcal{D})$ is again local. Since i is fully faithful, the limit cone of $i \circ p$ factors through i and defines a limit cone also in $\mathcal{C}$.
See also [1, Corollary 5.5.2.4] and [2, Corollary 4.1.5]. □

3. Adjoint Functor Theorems

Proposition 6. *Let $\mathcal{C}$ be a presentable ∞-category and let $F : \mathcal{C}^{op} \to \mathcal{S}$ be a functor. Then the following are equivalent:*

(1) F is representable.
(2) F preserves small limits.

Proof. See [1, Proposition 5.5.2.2].
(1)⇒(2): F is identified with the Yoneda embedding $\mathcal{C}^{op} \to \mathcal{P}(\mathcal{C}^{op})$ followed by the evaluation functor at the representing object. Both of these functors preserve all small limits.
(2)⇒(1): We first treat the case $\mathcal{C} = \mathcal{P}(\mathcal{D})$ for some small ∞-category $\mathcal{D}$. Let

$$(f : \mathcal{D}^{op} \xrightarrow{j^{op}} \mathcal{P}(\mathcal{D})^{op} \xrightarrow{F} \mathcal{S}) \in \mathcal{P}(\mathcal{D})$$

Then F is agrees with $\operatorname{map}(\_, f)$, as both functors preserve small limits and their restrictions to $\mathcal{D}$ agree by the Yoneda Lemma.

Now let $\mathcal{C}$ be an arbitrary presentable ∞-category. Let $L : \mathcal{P}(\mathcal{D}) \to \mathcal{C}$ be a localization as in Theorem 1 (5). Let S be the collections of morphisms in $\mathcal{P}(\mathcal{D})$ which map to equivalences under L. We may assume that $\mathcal{C}$ is the full subcategory of S-local objects. The composite functor

$$\mathcal{P}(\mathcal{D})^{\mathrm{op}} \xrightarrow{L^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{S}$$

preserves small limits and therefore it is represented by some object $f \in \mathcal{P}(\mathcal{D})$. Then it suffices to show that f is S-local. This holds because the representable functor FL^{op} factors through $\mathcal{C}^{\mathrm{op}}$. □

We also have the following dual version of the above proposition. This is a special case of Theorem 8(2) below. See also [1, Proposition 5.5.2.7].

Proposition 7. *Let $\mathcal{C}$ be a presentable ∞-category and let $F : \mathcal{C} \to \mathcal{S}$ be a functor. Then the following are equivalent:*

(1) F is corepresentable.
(2) F preserves limits and is accessible.

Theorem 8. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then:*

(1) F has a right adjoint $\Leftrightarrow$ F preserves small colimits.
(2) F has a left adjoint $\Leftrightarrow$ F preserves small limits and is accessible.

Proof. See [1, Corollary 5.5.2.9] and [2, Section 4].

(1) "$\Rightarrow$": This is always true [1, Proposition 5.2.3.5]. "$\Leftarrow$": Using the characterization in Subsection 1.1, it suffices to prove that for every representable functor $\mathcal{D}^{op} \to \mathcal{S}$, the composite $\mathcal{C}^{\mathrm{op}} \xrightarrow{F^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$ is again representable. This follows easily from Proposition 6.

(2) "$\Rightarrow$": Right adjoints preserve small limits (see [1, Proposition 5.2.3.5]). F is also accessible by Proposition 2.
"$\Leftarrow$": (see [2, Section 4]) According to Subsection 1.1, it suffices to show that for every $d \in \mathcal{D}$, the ∞-category $\mathcal{C}_{d/}$ has a weakly initial set, given that the following two assertions hold:
- $\mathcal{C}_{d/}$ is locally small as it is a pullback of locally small categories.
- $\mathcal{C}_{d/}$ is complete as it is a (homotopy) pullback of limit-preserving functors between complete categories (see Corollary 5 and apply [1, Lemma 5.4.5.5]).

Let κ be a regular cardinal such that $\mathcal{C}$ is κ-accessible, F is κ-continuous and d is κ-compact. Let $\mathcal{C}' \subseteq \mathcal{C}^{\kappa}$ be a small full subcategory such that the inclusion is an equivalence. Let $S :\, = \{(c, \alpha : d \to F(c)) \mid c \in \mathcal{C}'\}$ be a set of objects in $\mathcal{C}_{d/}$. Using the fact that every object in $\mathcal{C}$ is κ-filtered colimit of κ-compact objects and our assumptions on κ, it follows that for every $(c, d \to F(c))$ in $\mathcal{C}_{d/}$, there is a morphism from an object in S.

□

Proof. (of Proposition 7): *(1)$\Rightarrow$(2)*: Suppose that F is corepresented by the object $c \in \mathcal{C}$. Then F is identified with the composition $\mathcal{C} \xrightarrow{j} \mathcal{P}(\mathcal{C}) \xrightarrow{ev_c} \mathcal{S}$ and therefore it preserves limits. Moreover, we may choose a κ such that c is κ-compact, which then means that F preserves κ-filtered colimits. *(2)$\Rightarrow$(1)*: By Theorem 8(2), F admits a left adjoint L. Then there are canonical equivalences:

$$F(x) \simeq \mathrm{map}_{\mathcal{S}}(*, F(x)) \simeq \mathrm{map}_{\mathcal{C}}(L(*), x).$$

Therefore F is corepresented by the object $L(*)$. □

References

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