PRESENTABLE ∞-CATEGORIES – II

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1. Saturated Collections of Morphisms

Definition 1.1. Let $\mathcal{C}$ be an ∞-category and let S be a collection of morphisms in $\mathcal{C}$. We call S *(strongly) saturated* if

(1) S is closed under small colimits in $\mathrm{Fun}(\Delta^1, \mathcal{C})$.
(2) – (saturated) S contains all equivalences and is closed under composition.
    – (strongly saturated) S has the 2-out-of-3 property.
(3) S is closed under pushouts in $\mathcal{C}$.

Clearly the intersection of (strongly) saturated collections of morphisms is again (strongly) saturated. Therefore for every collection S of morphisms in $\mathcal{C}$, there is a smallest (strongly) saturated collection $S \subseteq \overline{S}$, called the *(strongly) saturated closure of* S. If for a (strongly) saturated collection S there is a small set S_0 such that $\overline{S}_0 = S$, we say that S is *of small generation*.

Remark 1.2. If $\mathcal{C}$ is cocomplete, then a strongly saturated collection S contains all equivalences, as these are pushouts of $\mathrm{id}_\emptyset$ in $\mathcal{C}$. In particular, S is saturated. Also, the collection of all equivalences in $\mathcal{C}$ is (strongly) saturated.

Remark 1.3. Let $\mathcal{C}, \mathcal{D}$ be cocomplete ∞-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor. Then the preimage of a (strongly) saturated collection S in $\mathcal{D}$ under the functor F is again (strongly) saturated.

Definition 1.4. Let $\mathcal{C}$ be a cocomplete ∞-category and S a collection of morphisms in $\mathcal{C}$. Then a morphism $f : x \to y$ in $\mathcal{C}$ is called a S*-equivalence* if for every S-local object $z \in \mathcal{C}$, the induced map:

$$\mathrm{map}(y, z) \xrightarrow{\sim} \mathrm{map}(x, z)$$

is an equivalence.

Lemma 1.5. *Let S be collection of morphisms in a cocomplete ∞-category $\mathcal{C}$. Then the collection of S-equivalences is strongly saturated.*

Proof. See [2, Lemma 5.5.4.11]. The collection of S-equivalences is the intersection of the preimages of equivalences under the functors $\mathrm{map}(\_, z)$ for every S-local object $z \in \mathcal{C}$. Note that these functors send colimits in $\mathcal{C}$ to limits. □

2. Localizations of Presentable ∞-Categories

Theorem 2.1. *Let $\mathcal{C}$ be presentable ∞-category, S a set of morphisms in $\mathcal{C}$, $\overline{S}$ its strongly saturated closure, and let $\mathcal{C}' \subseteq \mathcal{C}$ denote the full subcategory of $\overline{S}$-local objects. Then:*

(1) For every $C \in \mathcal{C}$, there exists a morphism $s : C \to C'$ in $\overline{S}$ where $C' \in \mathcal{C}'$.
(2) The inclusion $i \colon \mathcal{C}' \subseteq \mathcal{C}$ has a left adjoint $L \colon \mathcal{C} \to \mathcal{C}'$.

(3) For every morphism f in $\mathcal{C}$ the following are equivalent:
(a) f is an S-equivalence.
(b) $f \in \overline{S}$.
(c) Lf is an equivalence.
(4) $\mathcal{C}'$ is presentable.

Lemma 2.2. *Let $\mathcal{C}$ be a presentable ∞-category, S a saturated collection of morphisms in $\mathcal{C}$, and let $\mathcal{D} \subseteq Fun(\Delta^1, \mathcal{C})$ denote the full subcategory generated by S. The following are equivalent:*

(1) S is of small generation.
(2) The full subcategory $\mathcal{D} \subseteq Fun(\Delta^1, \mathcal{C})$ is presentable.

Proof. See [2, Lemma 5.5.5.9]. *(2)⇒(1)*: $\mathcal{D}$ is generated under small colimits by a (small) set of morphisms.

(1)⇒(2): The strategy of proof is different from the one in [2, Lemma 5.5.5.9] and is based on [1, Theorem 1(7)]. Let $S_0 \subseteq \mathcal{C}$ be a small collection of morphisms whose saturated closure is S. Let κ be a regular cardinal such that $\mathcal{C}$ is κ-accessible. Then the full subcategory of κ-compact objects $\mathcal{C}^\kappa$ is essentially small, and we may assume without loss of generality that S_0 contains the morphism id_x for every $x \in \mathcal{C}^\kappa$. Using one of the characterizations of presentable ∞-categories [1, Theorem 1], it suffices to show that S_0 detects equivalences. (Note that every object of $Fun(\Delta^1, \mathcal{C})$ is λ-compact for some λ and $\mathcal{D} \subseteq Fun(\Delta^1, \mathcal{C})$ is closed under colimits.)

Let $u : f_1 \to f_2$ be a morphism in $\mathcal{D}$ such that for every $s \in S_0$, the morphism

$$\mathrm{map}(s, u)\colon\ \mathrm{map}(s, f_1) \to \mathrm{map}(s, f_2)$$

is an equivalence. Then it suffices to show that this is also true for every $s \in S$. Consider the collection of morphisms

$$T := \{f \in S \mid \mathrm{map}(f, u) \text{ is an equivalence}\}.$$

We know that $S_0 \subseteq T$, therefore it suffices to show:

Claim. T is saturated.

Before we prove this claim, we first recall some **general facts** and make some useful observations:

(i) We can describe the mapping spaces in $\mathcal{C}^{\Delta^1}$ as follows: given $f, g \in \mathcal{C}^{\Delta^1}$ we have a canonical equivalence

(*) $$\mathrm{map}_{\mathcal{C}^{\Delta^1}}(f, g) \simeq \mathrm{map}_{\mathcal{C}}(\mathrm{dom}(f), \mathrm{dom}(g)) \underset{\mathrm{map}_{\mathcal{C}}(\mathrm{dom}(f), \mathrm{cod}(g))}{\times} \mathrm{map}_{\mathcal{C}}(\mathrm{cod}(f), \mathrm{cod}(g))$$

where the right space is a pullback in the ∞-category of spaces. This is can be seen by using the definition of the mapping space as a pullback of the diagram

$$\begin{array}{ccc} \mathrm{map}_{\mathcal{C}}(x, y) & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \downarrow & & \downarrow (\mathrm{dom}, \mathrm{cod}) \\ \Delta^0 & \xrightarrow{(x,y)} & \mathcal{C} \times \mathcal{C} \end{array}$$

together with the following (homotopy) pullback diagram:

$$\begin{array}{ccc} \operatorname{map}_{\mathcal{C}^{\Delta^1}}(h, \mathrm{id}_x) & \longrightarrow & \mathcal{C}^{\Delta^2} \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{(h,x)} & \mathcal{C}^{\Delta^1} \times \mathcal{C}. \end{array}$$

Then we obtain an equivalence:

$$\operatorname{map}_{\mathcal{C}^{\Delta^1}}(f, g) \simeq \operatorname{map}_{\mathcal{C}^{\Delta^1}}(f, \mathrm{id}_{\mathrm{cod}(g)}) \underset{\operatorname{map}_{\mathcal{C}}(\mathrm{dom}(f), \mathrm{cod}(g))}{\times} \operatorname{map}(\mathrm{id}_{\mathrm{dom}(f)}, g)$$

by looking at the fibres of the following equivalent morphisms:

$$(\mathcal{C}^{\Delta^1{}^{\Delta^1}} \longrightarrow \mathcal{C}^{\Delta^1} \times \mathcal{C}^{\Delta^1}) \simeq (\mathcal{C}^{\Delta^2} \times_{\mathcal{C}^{\Delta^1}} \mathcal{C}^{\Delta^2} \longrightarrow (\mathcal{C}^{\Delta^1} \times \mathcal{C}) \times_{\mathcal{C} \times \mathcal{C}} (\mathcal{C} \times \mathcal{C}^{\Delta^1})).$$

(ii) Let $f, g \in \mathcal{C}^{\Delta^1}$ and assume that f is an equivalence. Then composition with f induces an equivalence between mapping spaces:

$$\operatorname{map}_{\mathcal{C}}(\mathrm{cod}(f), \mathrm{cod}(g)) \xrightarrow{\simeq} \operatorname{map}_{\mathcal{C}}(\mathrm{dom}(f), \mathrm{cod}(g)).$$

So the equivalence (*) implies that restricting to the domain induces an equivalence

$$\text{(**)} \qquad \operatorname{map}_{\mathcal{C}^{\Delta^1}}(f, g) \simeq \operatorname{map}_{\mathcal{C}}(\mathrm{dom}(f), \mathrm{dom}(g)).$$

(iii) Applying (**) to the special case where $f = \mathrm{id}_x$ for $x \in \mathcal{C}$, we get

$$\operatorname{map}_{\mathcal{C}^{\Delta^1}}(\mathrm{id}_x, g) \simeq \operatorname{map}_{\mathcal{C}}(x, \mathrm{dom}(g))$$

and this equivalence is natural in g. Thus, since $\mathrm{id}_x \in S_0$ for every $x \in \mathcal{C}^\kappa$, the morphism $u : f_1 \to f_2$ induces an equivalence

$$\operatorname{map}_{\mathcal{C}}(x, \mathrm{dom}(f_1)) \simeq \operatorname{map}_{\mathcal{C}}(x, \mathrm{dom}(f_2))$$

for every $x \in \mathcal{C}^\kappa$. As $\mathcal{C}^\kappa$ detects equivalences in $\mathcal{C}$, it follows that u restricts to an equivalence between the domains:

$$\text{(***)} \qquad \mathrm{dom}(f_1) \simeq \mathrm{dom}(f_2)$$

Proof of the Claim. We can now prove that T is saturated:

(1) T is stable under small colimits in $\mathcal{C}^{\Delta^1}$: Let $F : K \to \mathcal{C}^{\Delta^1}$ be a diagram such that $F(k) \in T$ for every $k \in K$. We have an equivalence

$$\operatorname{map}_{\mathcal{C}^{\Delta^1}}(\operatorname{colim}_{k \in K}(F(k)), u) \simeq \lim_{k \in K}(\operatorname{map}_{\mathcal{C}^{\Delta^1}}(F(k), u)).$$

Thus, $\operatorname{map}_{\mathcal{C}^{\Delta^1}}(\operatorname{colim}_{k \in K}(F(k)), u)$ is a limit of equivalences and hence again an equivalence, so $\operatorname{colim}_{k \in K}(F(k))$ lies again in T.

(2) T contains the equivalences: For an equivalence f, the map $\operatorname{map}_{\mathcal{C}}(f, u)$ is by (**) equivalent to $\operatorname{map}_{\mathcal{C}}(\mathrm{dom}(f), \mathrm{dom}(u))$, which is an equivalence by (***).

(3) T is stable under pushouts: Consider a pushout in $\mathcal{C}$

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ x & \xrightarrow{f'} & y \end{array}$$

where f lies in T. For any $h \in \mathcal{C}^{\Delta^1}$, we have a natural pullback

$$\mathrm{map}_{\mathcal{C}}(y, \mathrm{cod}(h)) \simeq \mathrm{map}_{\mathcal{C}}(x, \mathrm{cod}(h)) \times_{\mathrm{map}_{\mathcal{C}}(a, \mathrm{cod}(h))} \mathrm{map}_{\mathcal{C}}(b, \mathrm{cod}(h)).$$

Combining this with the equivalence (*), we obtain natural equivalences:

$$\begin{aligned}
\mathrm{map}(f', h) &\overset{(*)}{\simeq} \mathrm{map}_{\mathcal{C}}(x, \mathrm{dom}(h)) \underset{\mathrm{map}_{\mathcal{C}}(x,\mathrm{cod}(h))}{\times} \mathrm{map}_{\mathcal{C}}(y, \mathrm{cod}(h)) \\
&\simeq \mathrm{map}_{\mathcal{C}}(x, \mathrm{dom}(h)) \underset{\mathrm{map}_{\mathcal{C}}(x,\mathrm{cod}(h))}{\times} (\mathrm{map}_{\mathcal{C}}(x, \mathrm{cod}(h)) \underset{\mathrm{map}_{\mathcal{C}}(a,\mathrm{cod}(h))}{\times} \mathrm{map}_{\mathcal{C}}(b, \mathrm{cod}(h))) \\
&\simeq \mathrm{map}_{\mathcal{C}}(x, \mathrm{dom}(h)) \underset{\mathrm{map}_{\mathcal{C}}(a,\mathrm{cod}(h))}{\times} \mathrm{map}_{\mathcal{C}}(b, \mathrm{cod}(h)) \\
&\simeq \mathrm{map}_{\mathcal{C}}(x, \mathrm{dom}(h)) \underset{\mathrm{map}_{\mathcal{C}}(a,\mathrm{dom}(h))}{\times} (\mathrm{map}_{\mathcal{C}}(a, \mathrm{dom}(h)) \underset{\mathrm{map}_{\mathcal{C}}(a,\mathrm{cod}(h))}{\times} \mathrm{map}_{\mathcal{C}}(b, \mathrm{cod}(h))) \\
&\overset{(*)}{\simeq} \mathrm{map}_{\mathcal{C}}(x, \mathrm{dom}(h)) \underset{\mathrm{map}_{\mathcal{C}}(a,\mathrm{dom}(h))}{\times} \mathrm{map}_{\mathcal{C}^{\Delta^1}}(f, h).
\end{aligned}$$

It follows that $\mathrm{map}(f', u)$ is an equivalence because it can be identified with the equivalence (this uses (***)):

$$\mathrm{map}_{\mathcal{C}}(x, \mathrm{dom}(f_1)) \underset{\mathrm{map}_{\mathcal{C}}(a,\mathrm{dom}(f_1))}{\times} \mathrm{map}_{\mathcal{C}^{\Delta^1}}(f, f_1) \simeq \mathrm{map}_{\mathcal{C}}(x, \mathrm{dom}(f_2)) \underset{\mathrm{map}_{\mathcal{C}}(a,\mathrm{dom}(f_2))}{\times} \mathrm{map}_{\mathcal{C}^{\Delta^1}}(f, f_2).$$

□

Lemma 2.3. *Left fibrations preserve colimits indexed by weakly contractible simplicial sets.*

Proof. See [2, Proposition 4.4.2.9]. □

Lemma 2.4. *Let $\mathcal{C}$ and S be as in Lemma 2.2 and let $X \in \mathcal{C}$ be an object. Then the full subcategory $\mathcal{D} \subseteq \mathcal{C}^{X/}$ spanned by the elements in S is closed under small colimits. (Here $\mathcal{C}^{X/} \simeq \mathcal{C}_{X/}$ denotes the alternative slice construction of [2, 4.2.1].)*

Proof. See [2, Lemma 5.5.5.11]. $\mathcal{D}$ admits κ-small colimits if and only if $\mathcal{D}$ admits finite colimits and κ-small filtered colimits by [2, Corollary 4.2.3.11]. It admits finite colimits if and only if it admits pushouts and has an initial object by [2, Corollary 4.4.2.4]. Hence it suffices to show that $\mathcal{D}$ contains an initial object of $\mathcal{C}^{X/}$ and is closed under colimits indexed by a small weakly contractible simplicial set K (take $K = \Lambda^2_0$ or K filtered).

(i) The initial objects in $\mathcal{C}^{X/}$ are the equivalences. These are contained in S by definition.

(ii) Consider a diagram $\overline{p} : K^{\triangleright} \to \mathcal{C}^{X/}$, which is a colimit of $p := \overline{p}|_K$, and p is a diagram in $\mathcal{D}$. By the construction of the alternative slice $\mathcal{C}^{X/}$, this corresponds to a map

$$P : K^{\triangleright} \times \Delta^1 \to \mathcal{C}$$

such that $P|_{K^{\triangleright} \times \{0\}} = \mathrm{const}_X$ and $P|_{K^{\triangleright} \times \{1\}} = (\mathcal{C}^{X/} \to \mathcal{C}) \circ \overline{p}$. Both of these two restrictions are colimit diagrams: The first one because $K \subseteq K^{\triangleright}$ is left anodyne, and thus cofinal, and the second one by Lemma 2.3. It follows that P is a colimit diagram in $Fun(\Delta^1, \mathcal{C})$. Since S is closed under colimits, we conclude that $\overline{p}$ is also a diagram in $\mathcal{D}$. □

Lemma 2.5. *Let $\mathcal{C}$ and S be as in Lemma 2.2 and assume that S is of small generation. Then for every object $X \in \mathcal{C}$, there exists a morphism $t : X \to Y$ in S such that Y is S-local.*

PROOF. See [2, Lemma 5.5.5.14]. Let $\mathcal{D} \subseteq Fun(\Delta^1, \mathcal{C})$ be the full subcategory spanned by S. Consider the pullback defined by

$$\begin{array}{ccc} \mathcal{D}_X & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{X} & Fun(\{0\}, \mathcal{C}) \end{array} \quad \text{or equivalently by} \quad \begin{array}{ccc} \mathcal{D}_X & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{C}_{X/} & \longrightarrow & \mathcal{C}^{\Delta^1} \end{array}$$

The ∞-category $\mathcal{D}_X \subseteq \mathcal{C}_{X/}$ is cocomplete (similarly to the proof of Lemma 2.4).

Claim. The ∞-category $\mathcal{D}_X$ is accessible.

Proof of the Claim. The functor $\mathcal{D} \to Fun(\{0\}, \mathcal{C})$ is a categorical fibration. To see this, consider a lifting problem:

$$\begin{array}{ccc} \{0\} & \xrightarrow{f} & \mathcal{D} \\ \downarrow & \nearrow & \downarrow \\ \Delta^1 & \xrightarrow{g} & \mathcal{C} \end{array} \quad \leftrightarrow \quad \begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ \downarrow g & & \vdots \\ \bullet & \cdots\rightarrow & \bullet \end{array} \in \mathcal{D}^{\Delta^1}$$

Then we get a diagram as in the right square by taking the pushout in $\mathcal{C}$. If g is an equivalence, then the resulting square defines an equivalence in $\mathcal{C}^{\Delta^1}$. Note that $\Delta^0, \mathcal{C}$ and $\mathcal{D}$ are accessible by assumption and Lemma 2.2. Moreover, the functor $\mathcal{D} \to \mathcal{C}$ is accessible, as S is closed under colimits in $\mathcal{C}^{\Delta^1}$. The functor $X \colon \Delta^0 \to \mathcal{C}$ is also accessible, as κ-filtered colimits are weakly contractible (and using the same argument as in the proof of Lemma 2.4). Then we conclude that $\mathcal{D}_X$ is accessible using the left pullback square above and [2, Proposition 5.4.6.6]. $\square$

Therefore $\mathcal{D}_X$ is presentable, and hence has a terminal object $X \xrightarrow{t} Y$. Then it remains to show that Y is S-local. Let $f : A \to B$ be a morphism in S. For any $g \colon A \to Y$, we form the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow g' \\ Y & \xrightarrow{f'} & Z \end{array}$$

and obtain pullback squares:

$$\begin{array}{ccc} \mathrm{map}_{\mathcal{C}_{A/}}(f, g) & \longrightarrow & \Delta^0 \\ \downarrow & & \downarrow \mathrm{id}_Y \\ \mathrm{map}_{\mathcal{C}}(Z, Y) & \xrightarrow{f'^*} & \mathrm{map}_{\mathcal{C}}(Y, Y) \\ \downarrow g'^* & & \downarrow g^* \\ \mathrm{map}_{\mathcal{C}}(B, Y) & \xrightarrow{f^*} & \mathrm{map}_{\mathcal{C}}(A, Y) \end{array} \qquad (*)$$

Hence $\mathrm{map}_{\mathcal{C}_{A/}}(f,g) \simeq \mathrm{map}_{\mathcal{C}_{Y/}}(f', \mathrm{id}_Y)$. It suffices to show that this mapping space is contractible. Consider a 2-simplex σ in $\mathcal{C}$ witnessing that a morphism $t' : X \to Z$ is the compostion of t and f'. Then we have a pullback:

$$\begin{array}{ccc}
\mathrm{map}_{\mathcal{C}_{Y/}}(f', \mathrm{id}_Y) \simeq \ \mathrm{map}_{\mathcal{C}_{t/}}(\sigma, \mathrm{id}_t) & \longrightarrow & \Delta^0 \\
\downarrow & & \downarrow \mathrm{id}_t \\
\mathrm{map}_{\mathcal{C}_{X/}}(t', t) & \xrightarrow{\sigma^*} & \mathrm{map}_{\mathcal{C}_{X/}}(t, t).
\end{array}$$

The morphism t' is in S because it is a composition of an element in S with the pushout of an element in S. Therefore the lower mapping spaces are mapping spaces in $\mathcal{D}_X$, where t is terminal. Hence σ^* is an equivalence and thus $\mathrm{map}_{\mathcal{C}_{A/}}(f,g) \simeq \mathrm{map}_{\mathcal{C}_{Y/}}(f', \mathrm{id}_Y)$ is contractible. So, the lowest horizontal map in diagram $(*)$ is an equivalence as required. □

Proof. (of Theorem 2.1) See [2, Proposition 5.5.4.15]. (1) follows from Lemma 2.5. (2): For $C \in \mathcal{C}$, we set $LC \coloneqq C' \in \mathcal{C}'$ and $t_C \colon C \to iLC$ in $\overline{S}$ (Lemma 2.5). Then for every $D \in \mathcal{D}$:

$$\mathrm{map}_{\mathcal{C}'}(LC, D) \simeq \mathrm{map}_{\mathcal{C}}(iLC, iD) \simeq \mathrm{map}_{\mathcal{C}}(C, iD)$$

where the last equivalence comes from the fact that D is $\overline{S}$-local. This property suffices to construct a left adjoint $L \colon \mathcal{C} \to \mathcal{C}'$ sending C to LC.

(3) "$(b) \Rightarrow (a)$": By Lemma 1.5 it suffices to see that $S \subseteq \{S - equivalences\}$, which is clear. (3)"$(c) \Rightarrow (b)$": Consider the diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
LX & \xrightarrow{Lf} & LY
\end{array}$$

The vertical maps lie in $\overline{S}$ by the construction of L in (2). By the 2-out-of-3 property, f lies in $\overline{S}$. (3) "$(a) \Rightarrow (c)$": We consider the same diagram:

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
LX & \xrightarrow{Lf} & LY
\end{array}$$

The vertical maps are in $\overline{S}$ and therefore they are S-equivalences by $(b) \Rightarrow (a)$. By the 2-out-of-3 property, Lf is also an S-equivalence. But then the natural transformation between the corepresentable functors:

$$Lf^* \colon \mathrm{map}_{\mathcal{C}'}(LY, \_) \to \mathrm{map}_{\mathcal{C}'}(LX, \_)$$

is an equivalence, and therefore $Lf \colon LX \to LY$ is an equivalence by the Yoneda Lemma.

(4): We outline a proof which is different from the proof in [2, Proposition 5.5.4.15(2)]. Let X be an S-local object. Then for any S-equivalence s, the map $\mathrm{map}(s, X)$ is an equivalence. By (3), X is also $\overline{S}$-local. This proves that $\mathcal{C}'$ is the full subcategory of S-local objects. It is thus the following pullback:

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{S}^{\simeq} \\ \big\downarrow & & \big\downarrow \psi \\ \mathcal{C} & \xrightarrow{\ \phi\ } & \mathcal{S}^{\Delta^1} \end{array}$$

where $\mathcal{S}$ denotes the ∞-category of spaces, ψ is the inclusion of the equivalences in $\mathcal{S}$, and ϕ is defined by

$$\phi(X) = \coprod_{(A \to B) \in S} (\mathrm{map}(B, X) \to \mathrm{map}(A, X))$$

which is well-defined because S is a set of morphisms. Note that the map $\mathcal{S} \to \mathcal{S}^{\simeq}$ sending x to id_x is an equivalence. To summarize, we have:

(i) $\mathcal{C}, \mathcal{S}^{\Delta^1}$ and $\mathcal{S}^{\simeq}$ are presentable ∞-categories.
(ii) As S is a set of morphisms, can choose a cardinal κ, such that the domain and the codomain of every morphism in S is κ-compact in $\mathcal{C}$. Then ϕ is κ-accessible.
(iii) ψ preserves colimits as these are computed pointwise in $\mathcal{S}^{\Delta^1}$.

Then it follows that $\mathcal{C}'$ is accessible [2, Proposition 5.4.6.6]. In particular, it is an accessible localization of a presentable ∞-category and therefore it is again presentable. □

Proposition 2.6. *Let $\mathcal{C}$ be a presentable ∞-category, S a set of morphisms, and $L : \mathcal{C} \to S^{-1}\mathcal{C}$ the localization functor of Theorem 2.1. Let $\mathcal{D}$ be an ∞-category. Then $L^* : Fun^L(S^{-1}\mathcal{C}, \mathcal{D}) \to Fun^L(\mathcal{C}, \mathcal{D})$ is fully faithful and its essential image consists of those $f : \mathcal{C} \to \mathcal{D}$ sending S to equivalences in $\mathcal{D}$. (Here $Fun^L(-,-)$ denotes the ∞-category of functors which preserve small colimits.)*

Proof. See [2, Proposition 5.5.4.20]. The functor is the restriction of the functor $L^* : Fun(S^{-1}\mathcal{C}, \mathcal{D}) \to Fun(\mathcal{C}, \mathcal{D})$, which is fully faithful, as $S^{-1}\mathcal{C}$ is a localization. Hence it is again fully faithful and we are left to show the part about the essential image, i.e., $\mathrm{im}(L^*) = \{F \in Fun^L(\mathcal{C}, \mathcal{D}) \mid \forall s \in S,\ f(s) \text{ is an equivalence}\}$.

"$\subseteq$": This is clear, as the functor L carries S to equivalences in $S^{-1}\mathcal{C}$.

"$\supseteq$": Let $F : \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor which carries S to equivalences in $\mathcal{D}$. The unit of the adjunction $\alpha : \mathrm{id} \to iL$ induces a natural transformation $F \to F \circ (iL)$. We will be done if we show that this natural transformation is an equivalence.
Let $S' := \{\phi \mid F(\phi) \text{ is an equivalence}\}$. This is strongly saturated by Remark 2 and also contains S. Therefore $\overline{S} \subseteq S'$. Since $\alpha_X \in \overline{S}$ for every object X in $\mathcal{C}$, it follows that $F(\alpha_X)$ is an equivalence as required.

□

3. Factorization Systems

Definition 3.1. Let $f : A \to B$ and $g : X \to Y$ be two morphisms in an ∞-category $\mathcal{C}$. We say that f is *orthogonal* to g (and write $f \perp g$) if for every diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & Y \end{array}$$

The space of lifting solutions $\mathrm{map}_{\mathcal{C}_{A//Y}}(B, X)$ is contractible.

Lemma 3.2. *Let $f : A \to B$ and $g : X \to Y$ be two morphisms in an ∞-category $\mathcal{C}$. Then the following are equivalent:*

(1) $f \perp g$.
(2) For every morphism $\overline{f} : \alpha \to \beta$ in $\mathcal{C}_{/Y}$ whose underlying morphism in $\mathcal{C}$ is f, the induced map

$$\mathrm{map}_{\mathcal{C}_{/Y}(\beta,g)} \xrightarrow{\overline{f}^*} \mathrm{map}_{\mathcal{C}_{/Y}(\alpha,g)}$$

is an equivalence.

Proof. See [2, Remark 5.2.8.3]. Given a diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{\beta} & Y \end{array}$$

we can restrict to the triangle $\overline{f}$:

$$\begin{array}{ccc} A & & \\ \downarrow{\scriptstyle f} & \searrow^{\alpha} & \\ B & \xrightarrow{\beta} & Y \end{array}$$

and then observe that $\mathrm{map}_{\mathcal{C}_{A//Y}}(B, X)$ is the homotopy fibre of the map in (2). □

Definition 3.3. Let $\mathcal{C}$ be an ∞-category and let S_L, S_R be collections of morphisms in $\mathcal{C}$. We say that (S_L, S_R) is a *factorization system* if the following are satisfied:

(1) S_L and S_R are closed under retracts.
(2) $S_L \perp S_R$.
(3) For every $h : X \to Z$ in $\mathcal{C}$, there is a factorization $h : X \xrightarrow{f} Y \xrightarrow{g} Z$ with $f \in S_L$ and $g \in S_R$.

Exercise 3.4. Let (S_L, S_R) be a factorization system in $\mathcal{C}$. Show that S_L is saturated. (Hint: Use Lemma 3.2.)

Given a collection S of morphisms in $\mathcal{C}$, we denote by $S^\perp$ the collection of the morphisms in $\mathcal{C}$ which are orthogonal to every morphism in S,

$$S^\perp := \{g \colon X \to Y \mid \forall s \in S : s \perp g\}.$$

Theorem 3.5. *Let $\mathcal{C}$ be a presentable ∞-category and S a saturated collection of small generation in $\mathcal{C}$. Then $(S, S^{\perp})$ is a factorization system in $\mathcal{C}$.*

Lemma 3.6. *Under the assumptions of Theorem 3.5, we have that for every $X \in \mathcal{C}$, the collection of morphisms in $\mathcal{C}_{/X}$*

$$S_X \coloneqq (\mathcal{C}_{/X} \to \mathcal{C})^{-1}(S)$$

is also a saturated collection of small generation.

Proof. See [2, Lemma 5.5.5.10]. S_X is again saturated by Remark 1.3. Consider the pullback

$$\begin{array}{ccc} \mathcal{D}' & \longrightarrow & \mathcal{C}_{/X}^{\Delta^1} \\ \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{C}^{\Delta^1} \end{array}$$

where $\mathcal{D}$ denotes the full subcategory of $\mathcal{C}^{\Delta^1}$ spanned by S. Then $\mathcal{D}'$ is the full subcategory of $\mathcal{C}_{/X}^{\Delta^1}$ spanned by S_X. Applying Lemma 2.2, we only need to show that $\mathcal{D}'$ is presentable. Using the properties of pullbacks with respect to cocompleteness and accessibility (see [2, Proposition 5.5.3.12]), this is then a consequence of the following observations:

(i) $\mathcal{D}$ is presentable by Lemma 2.2. $\mathcal{C}^{\Delta^1}$ and $\mathcal{C}_{/X}^{\Delta^1}$ are presentable because $\mathcal{C}$ is (see [2, Proposition 5.5.3.6] and [2, Proposition 5.5.3.10]).
(ii) $\mathcal{D} \to \mathcal{C}^{\Delta^1}$ preserves small colimits, as S is saturated. $\mathcal{C}_{/X}^{\Delta^1} \to \mathcal{C}^{\Delta^1}$ preserves small colimits. □

Proof. (of Theorem 3.5) See [2, Proposition 5.5.5.7]. Both collections of morphisms are closed under retracts, so (1) ✓. Clearly $S \perp S^{\perp}$, so (2) ✓. For (3), let $h : X \to Z$ be a morphism in $\mathcal{C}$. Then finding a factorization $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $f \in S$ and $g \in S^{\perp}$ is the same as finding a morphism $\overline{f} : h \to g$ in $\mathcal{C}_{/Z}$ such that $\overline{f} \in S_{\mathcal{Z}}$ and $g \in S^{\perp}$. Furthermore, using Lemma 3.2, we deduce that $g \in S^{\perp}$ if and only if g is S_Z-local. So we get the desired factorization by applying Lemma 2.5 to $(\mathcal{C}_{/Z}, S_Z)$, which we can do by Lemma 3.6, so also (3)✓. □

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