SPECTRUM OBJECTS AND STABILIZATION

SEBASTIAN WOLF

1. Stabilization

The goal of the following is to associate a stable ∞-category to any ∞-category with finite limits in a universal way. For this we will follow [2, section 1.4.2].

Definition 1.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between two ∞-categories where $\mathcal{D}$ has finite limits. Then we define:

i) If $\mathcal{C}$ has pushouts, then F is called *excisive* if it takes pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$.
ii) If $\mathcal{C}$ has a final object, then F is called *reduced* if F takes final objects to final objects.

We will write $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ (resp. $\mathrm{Exc}(\mathcal{C}, \mathcal{D})$) for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the reduced (resp. excisive) functors. We will also write $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ for the intersection $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D}) \cap \mathrm{Exc}(\mathcal{C}, \mathcal{D})$.

Remark 1.2. Let $\mathcal{C}$ be an ∞-category with finite colimits and a final object and $\mathcal{D}$ an ∞-category with finite limits. Then since limits commute with limits and limits in functor ∞-categories are computed objectwise, it follows that the ∞-categories $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$, $\mathrm{Exc}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ all again have finite limits.

Definition 1.3.

i) Let $\mathcal{C}$ be an ∞-category with a final object. Then we will write $\mathcal{C}_*$ for the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those arrows $f : x \to y$ in $\mathcal{C}$ for which x is a final object of $\mathcal{C}$. Note that this is equivalent to the slice category $\mathcal{C}_{1/}$ for a final object 1.
ii) Let $\mathcal{S}$ denote the ∞-category of spaces. Define $\mathcal{S}^{\mathrm{fin}}$ to be the smallest full subcategory of $\mathcal{S}$ that is closed under finite colimits and contains the point. We call $\mathcal{S}^{\mathrm{fin}}$ the ∞-category of *finite spaces*.
iii) For any $n \in \mathbb{N}$ we will define the *n-sphere* to be $S^n := \Sigma^n(S^0) \in \mathcal{S}^{fin}$. Here Σ denotes the suspension functor and $S^0 = * \coprod *$.
iv) For $\mathcal{C}$ an ∞-category with finite limits, we define the category of *spectrum objects* $\mathrm{Sp}(\mathcal{C}) := \mathrm{Exc}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$ and call an object $X \in \mathrm{Sp}(\mathcal{C})$ a *spectrum object* of $\mathcal{C}$.

Remark 1.4.

i) The category $\mathcal{S}^{\mathrm{fin}}$ has the following universal property. For any ∞-category $\mathcal{D}$ with finite colimits, evaluation at the point induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{rex}}(\mathcal{S}^{\mathrm{fin}}, \mathcal{D}) \simeq \mathcal{D}$. This follows from [1, Rem. 5.3.5.9] and [1, Prop. 4.3.2.15].
ii) For $K \in \mathrm{sSet}$, we obtain an isomorphism of ∞-categories $\mathrm{Sp}(\mathrm{Fun}(K, \mathcal{C})) \cong \mathrm{Fun}(K, \mathrm{Sp}(\mathcal{C}))$ induced by the canonical isomorphism $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathrm{Fun}(K, \mathcal{C})) \cong \mathrm{Fun}(K, \mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}, \mathcal{C}))$.
iii) Similiarly one can show that for a pointed ∞-category $\mathcal{C}$ with finite colimits and $\mathcal{D}$ an ∞-category with finite limits one has $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}_*) \cong \mathrm{Exc}_*(\mathcal{C}, \mathcal{D})_*$. Thus the following lemma shows that we get a canonical equivalence

$$\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$$

given by composing with the canonical projection $\mathcal{D}_* \to \mathcal{D}$

Lemma 1.5. *Let C be a pointed ∞-category with finite colimits, and $\mathcal{D}$ an ∞-category with finite limits. Then $\mathrm{Exc}_*(C, \mathcal{D})$ is pointed and has finite limits.*

Proof: The following is taken from [2, Lemma 1.4.2.10]. We know that $\mathrm{Exc}_*(C, \mathcal{D})$ has finite limits by remark 1.2. So it remains to show that $\mathrm{Exc}_*(C, \mathcal{D})$ is pointed. Let $*$ be a zero object of C and $*'$ a final object of $\mathcal{D}$. Denote by $X : C \to \mathcal{D}$ the constant functor at $*'$. Again since limits in functor categories are computed objectwise, X is a final object of $\mathrm{Exc}_*(C, \mathcal{D})$. Now one observes that the functor X is the left Kan extension of $*' : \{*\} \to \mathcal{D}$ along the inclusion $\{*\} \subseteq C$. Thus for any reduced excisive functor $\mathcal{F} \in \mathrm{Exc}_*(C, \mathcal{D})$ we get an equivalence of mapping spaces

$$\mathrm{map}_{\mathrm{Fun}(C,\mathcal{D})}(X, \mathcal{F}) \simeq \mathrm{map}_{\mathcal{D}}(X(*), \mathcal{F}(*)) \simeq *$$

since $\mathcal{F}(*) \simeq *'$. So X is a zero object of $\mathrm{Exc}_*(C, \mathcal{D})$ □

Our next goal is to show that the ∞-category $\mathrm{Exc}_*(C, \mathcal{D})$ is in fact stable. For this we will have to introduce some terminology first.

Notation 1.6. Let $F : C \to \mathcal{D}$ be a functor. Then for any commutative square τ

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

we obtain a canonical map $\eta_\tau : F(W) \to F(Y) \times_{F(z)} F(X)$. If C is pointed and has finite colimits and F is reduced, we get a map $F(X) \to F(*) \times_{F(\Sigma X)} F(*) \simeq * \times_{F(\Sigma X)} *$ induced by the suspension square

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

and we will denote this map by just η_X.

We now need the following technical lemma:

Lemma 1.7. *Let C be a pointed ∞-category with finite colimits, $\mathcal{D}$ an ∞-category with finite limits and $F : C \to \mathcal{D}$ a reduced functor. Suppose that we are given a pushout square τ*

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

in C. Then there exists a map $\theta : F(Y) \times_{F(Z)} F(X) \to * \times_{F(\Sigma W)} *$ such that:

i) $\theta \circ \eta_\tau \simeq \eta_W$

ii) *Let $\Sigma(\tau)$ denote the square*

$$\begin{array}{ccc} \Sigma W & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow \\ \Sigma Y & \longrightarrow & \Sigma Z. \end{array}$$

Then we get a pullback square

$$\begin{array}{ccc} \eta_{\Sigma(\tau)} \circ \theta & \longrightarrow & \eta_X \\ \downarrow & & \downarrow \\ \eta_Y & \longrightarrow & \eta_Z \end{array}$$

in $\mathrm{Fun}(\Delta^1, \mathcal{D})$.

Proof: Again we follow the proof given in [2, Lemma 1.4.2.15]. We have the following diagram in C

$$\begin{array}{ccccccc} W & \longrightarrow & X & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ Y & \longrightarrow & Z & \longrightarrow & Y \coprod_W 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \coprod_W X & \longrightarrow & \Sigma W & \longrightarrow & \Sigma Y \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \Sigma X & \longrightarrow & \Sigma Z \end{array}$$

in which every square is a pushout square. Applying F and replacing the upper left corner by a pullback we obtain the following diagram:

$$\begin{array}{ccccccc} F(Y) \times_{F(Z)} F(X) & \longrightarrow & F(X) & \longrightarrow & * & & \\ \downarrow & & \downarrow & & \downarrow & & \\ F(Y) & \longrightarrow & F(Z) & \longrightarrow & F(Y \coprod_W 0) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & F(0 \coprod_W X) & \longrightarrow & F(\Sigma W) & \longrightarrow & F(\Sigma Y) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & * & \longrightarrow & F(\Sigma X) & \longrightarrow & F(\Sigma Z) \end{array}$$

So we get an induced map $\theta : F(Y) \times_{F(Z)} F(X) \to * \times_{F(\Sigma W)} *$ by the universal property of the pullback and it is easy to see that this map satisfies i) and ii) above. □

We are now able to prove the following theorem which will be the main ingredient in order to show that $\mathrm{Exc}_*(C, \mathcal{D})$ is stable:

Theorem 1.8. *Let C be a pointed ∞-category with finite colimits and $\mathcal{D}$ an ∞- category with finite limits. Let $F : C \to \mathcal{D}$ be a reduced functor. Then the following are equivalent:*

i) *F is excisive.*
ii) *η_X is an equivalence for $X \in C$.*

Proof: (See [2, Prop. 1.4.2.13].) It is obvious that i) implies ii). For the converse let τ:

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

be a pushout square in C. It suffices to see that the map η_τ is an equivalence. By Lemma 1.7 there is a map $\theta : F(X) \times_{F(Z)} F(Y) \to * \times_{F(\Sigma W)} *$ such that $\theta \circ \eta_\tau \simeq \eta_W$ $(*)$. Since by assumption η_W is an equivalence, it suffices to see that θ is invertible. By (∗) we know that θ has a right inverse. Also by Lemma 1.7 we have a pullback square

$$\begin{array}{ccc} \eta_{\Sigma(\tau)} \circ \theta & \longrightarrow & \eta_X \\ \downarrow & & \downarrow \\ \eta_Y & \longrightarrow & \eta_Z \end{array}$$

in which η_X, η_Y and η_Z are equivalences, so $\eta_{\Sigma(\tau)} \circ \theta$ is an equivalence as well and so θ has a right inverse and is thus invertible. □

Remark 1.9. Note that theorem 1.9 is slightly stronger than the version in [2], since we dropped the assumption that $\mathcal{D}$ is pointed. However we have seen above that this assumption is not neeeded in the proofs. Alternatively one could also deduce theorem 1.8 from the version in [2] by using the equivalence

$$\mathrm{Exc}_*(C, \mathcal{D}_*) \to \mathrm{Exc}_*(C, \mathcal{D})$$

from the 1.4 iii) and the fact that the projection $\mathcal{D}_* \to \mathcal{D}$ creates finite limits by [1, Prop. 1.2.13.8]. Using this slightly more general version we can already deduce the following theorem ([2, Cor. 1.4.2.27]):

Theorem 1.10. *Let C be a pointed ∞- category. Then the following are equivalent:*

i) *C is stable.*
ii) *C has finite limits and $\Omega : C \to C$ is an equivalence.*
iii) *C has finite colimits and $\Sigma : C \to C$ is an equivalence.*

Proof: It is clear that ii) is equivalent to iii) by dualizing. It is also clear that i) implies ii). We will now show that ii) implies i). Consider for every object $x \in C$, the reduced functor $\mathrm{map}_C(-, x) : C^{\mathrm{op}} \to \mathcal{S}$. Let Σ denote an inverse of Ω. Note that by being an inverse of the loop functor Ω, the functor Σ automatically acts as a suspension functor, i.e. one has

$$\mathrm{map}_C(\Sigma a, b) \simeq \mathrm{map}_C(a, \Omega b) \simeq \Omega\, \mathrm{map}_C(a, b)$$

for any a and b in C. Note that the ∞-category C^{op} is pointed and has all finite colimits and $\mathcal{S}$ has all finite limits. For every $c \in C$ we have that $c \simeq \Sigma\Omega c$ and thus the induced map

$$\mathrm{map}_C(c, x) \simeq \mathrm{map}_C(\Sigma\Omega c, x) \simeq \Omega\, \mathrm{map}_C(\Omega c, x)$$

is an equivalence. Thus by theorem 1.8 the functor $\mathrm{map}_C(-, x)$ takes pushout squares in C^{op} to pullback squares in $\mathcal{S}$ for any $x \in C$. In other words, pullback squares in C are pushout squares. To complete the proof it suffices to show that C has cofibres. For any morphism $f : x \to y$ in C one has a diagram

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & * \\ \downarrow & & \downarrow \\ x & \xrightarrow{f} & y \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma\mathrm{fib}(f) \end{array}$$

where the top square and the outer square are both pushouts by what we have seen above. So the lower square is also a pushout and hence C has cofibres. □

Theorem 1.11. *Let C be an ∞-category with finite colimits and $\mathcal{D}$ an ∞-category with finite limits. Then $\mathrm{Exc}_*(C, \mathcal{D})$ is stable.*

Proof: We have already seen that $\mathrm{Exc}_*(C,\mathcal{D})$ is pointed and has finite limits, so by theorem 1.10 it suffices to see that the functor $\Omega := \Omega_{\mathrm{Exc}_*(C,\mathcal{D})}$ is an equivalence of ∞-categories. But one can easily check that the functor $\Sigma_C^* : \mathrm{Fun}(C,\mathcal{D}) \to \mathrm{Fun}(C,\mathcal{D})$, given by precomposition with Σ_C, restricts to $\mathrm{Exc}_*(C,\mathcal{D})$ and is an inverse to Ω. □

Remark 1.12. Note that here the proof of theorem 1.11 is a bit simpler than the one given in [2], because we already have 1.10 at hand and thus do not have worry about the existence of finite colimits in $\mathrm{Exc}_*(C,\mathcal{D})$.

Corollary 1.13. *Let C be an ∞-category with finite limits, then* $\mathrm{Sp}(C)$ *is stable.*

We will now work towards formulating and proving a universal property for $\mathrm{Sp}(C)$:

Proposition 1.14. *Let C be an ∞-category with finite colimits and a final object and $\mathcal{D}$ a stable ∞-category. Let $f : C \to C_*$ be a left adjoint to the forgetful functor $C_* \to C$. Denote by* $\mathrm{Exc}'(C,\mathcal{D})$ *the full subcategory of* $\mathrm{Exc}(C,\mathcal{D})$ *spanned by those functors which send the initial object of C to a terminal object of $\mathcal{D}$. Then composition with f induces an equivalence of ∞-categories $f^* : \mathrm{Exc}_*(C_*,\mathcal{D}) \to \mathrm{Exc}'(C,\mathcal{D})$.*

Proof: Again we follow the proof in [2, Lemma 1.4.2.19]. Consider the composite

$$\theta : \mathrm{Fun}(C,\mathcal{D}) \times C_* \subseteq \mathrm{Fun}(C,\mathcal{D}) \times \mathrm{Fun}(\Delta^1, C) \xrightarrow{\text{comp}} \mathrm{Fun}(\Delta^1,\mathcal{D}) \xrightarrow{\text{cofib}} \mathcal{D}.$$

One can then check that the transposed functor

$$\mathrm{Fun}(C,\mathcal{D}) \to \mathrm{Fun}(C_*,\mathcal{D})$$

restricts to a functor $\mathrm{Exc}'(C,\mathcal{D}) \to \mathrm{Exc}_*(C_*,\mathcal{D})$ which is an inverse of f^*. □

Notation 1.15. For C an ∞-category with finite limits, denote the evaluation functor $\mathrm{ev}_{S^0} : \mathrm{Sp}(C) \to C$ by Ω^∞. More generally, for any $n \in \mathbb{N}$ denote by $\Omega^{\infty-n}$ the composition $\Omega^\infty \circ \Sigma_C^n$.

Proposition 1.16. *Let $\mathcal{D}$ be an ∞-category with finite limits. Then the following are equivalent*

i) *$\mathcal{D}$ is stable.*
ii) *$\Omega^\infty : \mathrm{Sp}(C) \to C$ is an equivalence.*

Proof: (See [2, Prop. 1.4.2.21].) We have already seen that ii) implies i). So let us assume that $\mathcal{D}$ is stable. Then we obtain a homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{Sp}(\mathcal{D}) & \xrightarrow{\Omega^\infty} & \mathcal{D} \\ \simeq\downarrow & & \uparrow \mathrm{ev}_* \\ \mathrm{Exc}'(\mathcal{S}^{\mathrm{fin}},\mathcal{D}) & \xrightarrow{\simeq} & \mathrm{Fun}^{\mathrm{rex}}(\mathcal{S}^{\mathrm{fin}},\mathcal{D}) \end{array}$$

where the left vertical arrow is the equivalence of proposition 1.14 and the bottom arrow is an equivalence since $\mathcal{D}$ is stable. Now by remark 1.4 i), ev_* is an equivalence as well and thus we get our claim. □

We are now ready to prove that the category of spectrum objects has the following universal property:

Proposition 1.17. *Let C be a pointed ∞-category with finite colimits and $\mathcal{D}$ an ∞-category with finite limits. Then we get an equivalence of ∞-categories*

$$\Omega^\infty_* : \mathrm{Exc}_*(C,\mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_*(C,\mathcal{D})$$

given by composing with Ω^∞.

Proof: (See [2, Prop. 1.4.2.22].) We get a homotopy commutative square

$$\begin{array}{ccc} \mathrm{Exc}_*(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) & & \\ \downarrow \cong & \searrow^{\Omega^\infty_*} & \\ \mathrm{Sp}(\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})) & \xrightarrow[\Omega^\infty_{\mathrm{Exc}_*(\mathcal{C},\mathcal{D})}]{} & \mathrm{Exc}_*(\mathcal{C}, \mathcal{D}) \end{array}$$

where the bottom square is an equivalence by proposition 1.16 since $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ is stable by theorem 1.11, and the left vertical arrow is the canonical isomorphism given by restricting

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{D})) \cong \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathrm{Fun}(\mathcal{C}, \mathcal{D}))$$

so we get our claim. □

We will now produce an alternative description of the category of spectrum objects:

Theorem 1.18. *Let $\mathcal{C}$ be a pointed ∞-category with finite limits. Then the functor Ω^∞ : $\mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ induces an equivalence of $\mathrm{Sp}(\mathcal{C})$ with the limit of the tower of ∞-categories*

$$... \to \mathcal{C} \xrightarrow{\Omega_\mathcal{C}} \mathcal{C} \xrightarrow{\Omega_\mathcal{C}} \mathcal{C}$$

Proof: Let $\widetilde{\mathrm{Sp}}(\mathcal{C})$ denote the limit of the tower above. Then by construction $\widetilde{\mathrm{Sp}}(\mathcal{C})$ is pointed and has finite limits and the loop functor $\Omega^\infty_{\widetilde{\mathrm{Sp}}(\mathcal{C})}$ is an equivalence on $\widetilde{\mathrm{Sp}}(\mathcal{C})$, so by theorem 1.10 it is stable. Hence it suffices to see that for any stable ∞-category $\mathcal{D}$ the canonical functor $\phi : \mathrm{Sp}(\mathcal{C}) \to \widetilde{\mathrm{Sp}}(\mathcal{C})$ induced by $\Omega^{\infty-n}$ for all $n \in \mathbb{N}$, induces an equivalence of ∞-categories:

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathrm{Sp}(\mathcal{C})) \xrightarrow{\phi_*} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \widetilde{\mathrm{Sp}}(\mathcal{C})).$$

Again we obtain a homotopy commutative square

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathrm{Sp}(\mathcal{C})) & \xrightarrow{\phi_*} & \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \widetilde{\mathrm{Sp}}(\mathcal{C})) \\ \Omega^\infty_* \downarrow & \swarrow_{pr_{0,*}} & \\ \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathcal{C}). & & \end{array}$$

where $pr_0 : \widetilde{\mathrm{Sp}}(\mathcal{C}) \to \mathcal{C}$ denotes the canonical projection from the inverse limit. Now Ω^∞_* is an equivalence by theorem 1.17 as $\mathcal{D}$ is stable, and thus it suffices to see that $pr_{0,*}$ is an equivalence. But we have that

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \widetilde{\mathrm{Sp}}(\mathcal{C})) \simeq \lim(... \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathcal{C}) \xrightarrow{\Omega_*} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathcal{C}) \xrightarrow{\Omega_*} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}, \mathcal{C}))$$

and so it suffices to see that Ω_* is an equivalence (here Ω_* denotes the functor given by composition with Ω). But one can easily check that the functor given by precomposition with $\Sigma_\mathcal{C}$ is an inverse for Ω_*, which proves our claim. □

Remark 1.19. Again, the proof here is easier than the one given in [2], because we can use theorem 1.10 and do not have to worry about the existence of finite colimits in $\widetilde{\mathrm{Sp}}(\mathcal{C})$.

Example 1.20. We define the ∞-category of *Spectra* to be $\mathrm{Sp} := \mathrm{Sp}(\mathcal{S})$ and call an object $X \in \mathrm{Sp}(\mathcal{S})$ a *spectrum*. Combining remark 1.4 iii) and theorem 1.18 one sees that a spectrum can be identified with a sequence of pointed spaces $(X_n)_{n\in\mathbb{N}}$ together with identifications $\Omega X_{n+1} \simeq X_n$. In the classical literature these objects are usually referred to as Ω-spectra. In particular it follows that the homotopy category $h(\mathrm{Sp})$ is equivalent to the classical stable homotopy category.

2. Presentable Stable ∞-categories

In this section we will study ∞-categories which are both presentable and stable at the same time. We will follow [2, Sect. 1.4.4].

Proposition 2.1. *Let C and $\mathcal{D}$ be presentable ∞-categories and let $\mathcal{D}$ be stable. Then*

i) $\mathrm{Sp}(C)$ *is presentable.*
ii) *The functor* $\Omega^\infty : \mathrm{Sp}(C) \to C$ *admits a left adjoint* $\Sigma^\infty_+ : C \to \mathrm{Sp}(C)$.
iii) *An exact functor* $G : \mathcal{D} \to \mathrm{Sp}(C)$ *admits a left adjoint if and only if* $\Omega^\infty \circ G : \mathcal{D} \to C$ *admits a left adjoint.*

Proof. We follow the proof of [2, Prop. 1.4.4.4]. Let us start with ii). Observe that the composite

$$\mathrm{Sp}(C) \xrightarrow{\simeq} \mathrm{Sp}(C_*) \xrightarrow{\Omega^\infty_*} C_* \xrightarrow{\mathrm{pr}} C$$

can be identified with Ω^∞. By the adjoint functor theorem it suffices to check that Ω^∞ preserves all limits and is accessible. As the projection $\mathrm{pr} : C_* \to C$ creates limits and weakly contractible colimits by [1, Prop.1.2.13.8] and [1, Prop. 4.4.2.9] it suffices to see that Ω^∞_* admits a left adjoint. But since $\Omega_* : C_* \to C_*$ admits a left ajoint and C_* is also presentable by [1, Prop. 5.5.3.11], Ω_* can be considered as a morphism in $\mathcal{P}r^R$. Here $\mathcal{P}r^R$ denotes the ∞-category of presentable ∞-categories and right adjoint functors. Furthermore the inclusion $\mathcal{P}r^R \to \mathrm{Cat}_\infty$ preserves limits ([1, Thm. 5.5.3.18]). It follows that the projection

$$\lim(... \to C_* \xrightarrow{\Omega_*} C_*) \to C_*$$

admits a left adjoint as well, but by theorem 1.18, this functor can be identified with Ω^∞_*, which proves ii). In particular the above limit description shows that $\mathrm{Sp}(C)$ is presentable, so we get i).
For iii), it immediately follows from ii) that the existence of a left adjoint of G implies the existence of a left adjoint of $\Omega^\infty \circ G$. So let us assume that $\Omega^\infty \circ G$ admits a left adjoint. Again $\Omega^\infty \circ G$ is given by the composition

$$\mathcal{D} \xrightarrow{G} \mathrm{Sp}(C) \xrightarrow{\simeq} \mathrm{Sp}(C_*) \xrightarrow{\Omega^\infty_*} C_* \xrightarrow{\mathrm{pr}} C$$

and let us denote the composition $\mathcal{D} \to \mathrm{Sp}(C_*)$ by G'. The same arguments as above show that $\Omega^\infty_* \circ G'$ admits a left adjoint as well. Now it again follows from the limit description in theorem 1.18 and [1, Thm. 5.5.3.18] that G' and thus G admits a left adjoint as well. ◻

Corollary 2.2. *Let C and $\mathcal{D}$ be presentable ∞-categories and assume that $\mathcal{D}$ is stable. We denote by* $\mathrm{LFun}(C, \mathcal{D})$ *the full subcategory of* $\mathrm{Fun}(C, \mathcal{D})$ *spanned by those functors that are left adjoints. Then precomopisition with Σ^∞_+ induces an equivalence of ∞-categories*

$$\Sigma^{\infty,*}_+ : \mathrm{LFun}(\mathrm{Sp}(C), \mathcal{D}) \to \mathrm{LFun}(C, \mathcal{D}).$$

Proof: The statement is dual to Ω^∞ inducing an equivalence of ∞-categories

$$\Omega^\infty_* : \mathrm{RFun}(\mathcal{D}, \mathrm{Sp}(C)) \to \mathrm{RFun}(\mathcal{D}, C)$$

but this follows by combining proposition 1.17 and 2.1. ◻

Definition 2.3. We will call $S := \Sigma^\infty_+(*) \in \mathrm{Sp}(\mathcal{S})$ the *sphere spectrum*.

The following proposition shows that the category of spectra can be viewed as the analogue of the category of spaces in the stable world:

Corollary 2.4. *Let $\mathcal{D}$ be a presentable stable ∞-category. Then evaluation at S induces an equivalence of ∞-categories*

$$\mathtt{ev}_S : \mathrm{LFun}(\mathrm{Sp}, \mathcal{D}) \to \mathcal{D}$$

Proof: We have a homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{LFun}(\mathrm{Sp}, \mathcal{D}) & \xrightarrow{\Sigma_+^{\infty,*}} & \mathrm{LFun}(\mathcal{S}, \mathcal{D}) \\ & \searrow^{\mathtt{ev}_S} & \downarrow{\scriptstyle \mathtt{ev}_*} \\ & & \mathcal{D} \end{array}$$

and $\mathtt{ev}_*$ and $\Sigma_+^{\infty,*}$ are both equivalences. □

We will now try to provide characterizations of stable and presentable ∞-categories. For this we will need the following two lemmas

Lemma 2.5. *Let C be a stable ∞-category and $C' \subseteq C$ a localization of C. Let $L : C \to C'$ be a left adjoint to the inclusion. Then L is left exact if and only if C' is stable.*

Proof: (See [2, Lemma 1.4.4.7]). It is obvious that if C' is stable, then L is left exact. So let us assume that L is left exact. Since C' is closed under limits in C it suffices to see that C' is closed under pushouts by [2, Lemma 1.1.3.3]. So let τ:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ z & \longrightarrow & W \end{array}$$

be a pushout square in C, where X, Y and Z are in C'. Since C is stable τ is a pullback square and thus $L(\tau)$ is a pullback square in C as well since L is left exact. Again since C is stable $L(\tau)$ is also a pushout square in C. But since X, Y and Z are objects of C' the unit map $\tau \to L(\tau)$ induces an equivalence $T \xrightarrow{\eta_T} LT$ for $T \in \{X, Y, Z\}$. Thus the unit map $\eta_W : W \to LW$ is an equivalence as well, as both τ and $L(\tau)$ are pushout squares. It follows that W is in the essential image of the inclusion $C' \subseteq C$, which proves our claim. □

Lemma 2.6. *Let C be an ∞-category with finite limits, $\mathcal{D}$ a stable ∞-category and $G : \mathcal{D} \to \mathrm{Sp}(C)$ an exact functor. Then if $g := \Omega^\infty \circ G : \mathcal{D} \to C$ is fully faithful, G is itself fully faithful.*

Proof: (See [2, Lemma 1.4.4.8]). Invoking remark 1.4 iii) and theorem 1.18, it suffices to prove that the map

$$g_n : \mathcal{D} \xrightarrow{G} \mathrm{Sp}(C) \xrightarrow{\Omega_*^{\infty-n}} C_*$$

is fully faithful for all n. Furthermore, since G is exact, we have that $g_n \simeq g_{n+1} \circ \Omega_{\mathcal{D}}$, where $\Omega_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is the loop functor. Since $\mathcal{D}$ is stable, $\Omega_{\mathcal{D}}$ is an equivalence, so it suffices to check that g_0 is fully faithful. So let us pick $X, Y \in \mathcal{D}$. We have a homotopy pullback square

$$\begin{array}{ccc} \mathrm{map}_{C_*}(g_0(X), g_0(Y)) & \xrightarrow{\varphi} & \mathrm{map}_C(g(X), g(Y)) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{map}_C(*_C, g(Y)) \end{array}$$

where $*_C$ denotes as final object of C. Since by assumption g is fully faithful, it suffices to see that φ is an equivalence. Furthermore the square above is a pullback and so it suffices to see that the bottom right corner is contractible. But as g is fully faithful and exact we have that

$$\mathrm{map}_{\mathcal{C}}(*_{\mathcal{C}}, g(Y)) \simeq \mathrm{map}_{\mathcal{D}}(*_{\mathcal{D}}, Y)$$

where $*_{\mathcal{D}}$ denotes a final object of $\mathcal{D}$. The claim now follows since $\mathcal{D}$ is stable, because then $*_{\mathcal{D}}$ is also an initial object. □

We can now give the following characterization of stable presentable ∞-categories. Note that again the category of spectra plays the role that the category of spaces plays in the unstable case.

Theorem 2.7. *Let $\mathcal{C}$ be an ∞-category. Then the following are equivalent:*

i) *$\mathcal{C}$ is presentable and stable.*
ii) *There exists a presentable and stable ∞-category $\mathcal{D}$ and an accessible left exact localization functor $L : \mathcal{D} \to \mathcal{C}$.*
iii) *There exists a small ∞-category $\mathcal{E}$ and an accessible left exact localization functor $L : \mathrm{Fun}(\mathcal{E}^{op}, \mathrm{Sp}) \to \mathcal{C}$.*

Proof: Again we follow the proof in [2, Prop. 1.4.4.9]. Note that since limits and colimits in functor categories are computed objectwise, it is clear that $\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp})$ is again stable. It is presentable as well since Sp is presentable by proposition 2.1. Thus iii) implies ii). It is clear that ii) implies i) by lemma 2.5. We will now show that i) implies iii). Since $\mathcal{C}$ is presentable there exists a small ∞-category $\mathcal{E}$ and an accesible fully faithful embedding $i : \mathcal{C} \to P(\mathcal{E})$ which admits a left adjoint. By the dual of corollary 2.2 there exists a factorization

$$i : \mathcal{C} \xrightarrow{G} \mathrm{Sp}(P(\mathcal{E})) \xrightarrow{\Omega^\infty} P(\mathcal{E})$$

where G admits a left adjoint. By lemma 2.6, G is fully faithful and thus $\mathcal{C}$ is an accessible left exact localization of $\mathrm{Sp}(P(\mathcal{E})) \cong \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp})$. □

References

[1] Jacob Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, Vol. 170, Princeton University Press, Princeton, NJ, 2009. Online revised version available at `https://www.math.ias.edu/~lurie/`

[2] Jacob Lurie, *Higher Algebra*, Online version available at `https://www.math.ias.edu/~lurie/`