STABLE ∞-CATEGORIES

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1. Stable ∞-categories

Definition: Let $\mathscr{C}$ be an ∞-category. An object c of $\mathscr{C}$ is called a *zero object* if it is both initial and terminal. The ∞-category $\mathscr{C}$ is called *pointed* if it has a zero object.

Definition: Let $\mathscr{C}$ be a pointed ∞-category with zero object 0. A *triangle* in $\mathscr{C}$ is a square $\Delta^1 \times \Delta^1 \to \mathscr{C}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & C. \end{array}$$

A triangle is called a *fiber sequence* if it is a cartesian square, and a *cofiber sequence* if it is a cocartesian square.
If the above triangle is a fiber sequence, then A is called the *fiber* of g and is also denoted $fib(g)$.
If the above triangle is a cofiber sequence, then B is also called the *cofiber* of f and is denoted $cofib(f)$.

Definition: An ∞-category $\mathscr{C}$ is called *stable* if it satisfies the following conditions:
(1) $\mathscr{C}$ is pointed.
(2) Every morphism in $\mathscr{C}$ has a fiber and a cofiber.
(3) A triangle in $\mathscr{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Remark: Stability of an ∞-category is a *property*, rather than additional structure.

See [HA, 1.1.1].

2. The Homotopy Category of a Stable ∞-Category

The homotopy category of a stable ∞-category admits the structure of a triangulated category in a canonical way [HA, 1.1.2] We recall the definition of a triangulated category.

Definition: An ordinary (1−)category $\mathscr{A}$ is called *additive* if it is enriched over the category *Ab* of abelian groups and has finite biproducts.

Definition: A *triangulated category* consists of the following data:
(1) An additive category $\mathscr{A}$.
(2) A functor $T : \mathscr{A} \to \mathscr{A}$ called the *translation functor*.
(3) A collection of *distinguished triangles*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

such that the following axioms are satisfied:

(TR1) (a) Every morphism $f : X \to Y$ in $\mathscr{D}$ can be extended to a distinguished triangle in $\mathscr{D}$.
(b) The collection of distinguished triangles is stable under isomorphism.
(c) For every object X in $\mathscr{D}$, the triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow TX$$

is distinguished.

(TR2) A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} TA \xrightarrow{-Tf} TB$$

is distinguished.

(TR3) Given a commutative diagram in $\mathscr{A}$

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\ \downarrow{\scriptstyle f} & & \downarrow & & & & \downarrow{\scriptstyle Tf} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \end{array}$$

there exists a morphism $C \to C'$ rendering the diagram commutative.
(Note that this morphism is not required to be unique!)

(TR4) Given three distinguished triangles

$$A \xrightarrow{f} B \longrightarrow B/A \longrightarrow TA$$

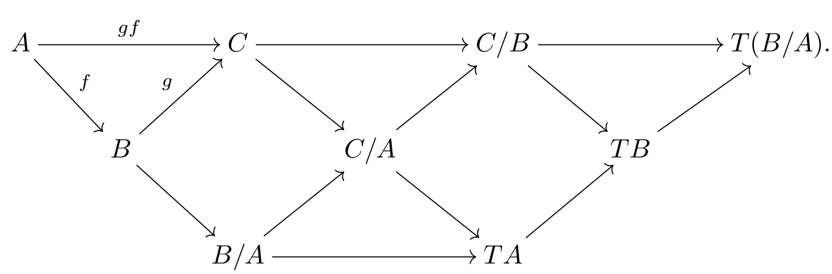
$$B \xrightarrow{g} C \longrightarrow C/B \longrightarrow TB$$

$$A \xrightarrow{gf} C \longrightarrow C/A \longrightarrow TA$$

there exists a fourth distinguished triangle

$$B/A \longrightarrow C/A \longrightarrow C/B \longrightarrow T(B/A)$$

making the following diagram commute:



Let $\mathscr{C}$ be a stable ∞-category with zero object 0. We define a triangulated structure on the homotopy category $h\mathscr{C}$ as follows. Let $\mathscr{M}$ be the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$ spanned by the cocartesian squares of the form

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C. \end{array}$$

The functor that takes such a square and sends it to the object A yields a trivial fibration $\mathscr{M} \to \mathscr{C}$. Let s denote a section of this map. Furthermore, let $e : \mathscr{M} \to \mathscr{C}$ be the functor that sends a square as above to the object C. Then the composition

$$e \circ s : \mathscr{C} \to \mathscr{C}$$

is called the *suspension functor* and is denoted by Σ. Dually, we can define the *loop functor* $\Omega : \mathscr{C} \to \mathscr{C}$ by sending an object C of $\mathscr{C}$ to the pullback of $(0 \to C \leftarrow 0)$.

The stability of $\mathscr{C}$ implies that Σ and Ω are mutually inverse equivalences $\mathscr{C} \to \mathscr{C}$. We define the translation functor $T : h\mathscr{C} \to h\mathscr{C}$ to be the functor that Σ induces on the homotopy category.

By the universal property of the pushout that defines Σ, we have an equivalence of mapping spaces

$$\mathrm{map}_{\mathscr{C}}(\Sigma(X), Y)) \xrightarrow{\simeq} \Omega \mathrm{map}_{\mathscr{C}}(X, Y)$$

which is natural in $X, Y \in \mathscr{C}$. Using this equivalence, we obtain an enrichment of $h\mathscr{C}$ in abelian groups. More specifically, given $X, Y \in \mathscr{C}$, let Z be an object such that $\Sigma^2 Z \simeq X$, then we have:

$$\mathrm{map}_{\mathscr{C}}(X, Y) \simeq \mathrm{map}_{\mathscr{C}}(\Sigma^2(Z), Y) \simeq \Omega^2 \mathrm{map}_{\mathscr{C}}(Z, Y),$$

so $\mathrm{Hom}_{h\mathscr{C}}(X, Y) = \pi_0 \mathrm{map}_{\mathscr{C}}(X, Y) \cong \pi_2 \mathrm{map}_{\mathscr{C}}(Z, Y)$ admits a canonical and natural abelian group structure. This makes $h\mathscr{C}$ *enriched over Ab*.

We next show that $h\mathscr{C}$ has *finite biproducts*. It suffices to show that it has finite coproducts – since in any category enriched over commutative monoids, finite coproducts are automatically biproducts. In fact, we will show the slightly stronger statement that $\mathscr{C}$ has finite coproducts. Let X and Y be objects in $\mathscr{C}$. Note that:

$$X \simeq \mathrm{cofib}(\Omega X \xrightarrow{u} 0) \text{ and } Y \simeq \mathrm{cofib}(0 \xrightarrow{v} Y).$$

The composite map $w : \Omega X \to 0 \to Y$ is a coproduct of u and v in $\mathrm{Fun}(\Delta^1, \mathscr{C})$. The functor cofib preserves all colimits, so $\mathrm{cofib}(w)$ is a coproduct of X and Y in $\mathscr{C}$. Hence $h\mathscr{C}$ has finite biproducts, and is thus additive.

Next we define the *distinguished triangles in* $h\mathscr{C}$. We say that a diagram

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

is a distinguished triangle if there is a diagram $\Delta^1 \times \Delta^2 \to \mathscr{C}$ of the form

$$\begin{array}{ccccc} A & \xrightarrow{f'} & B & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle g'} & & \downarrow \\ 0 & \longrightarrow & C & \xrightarrow{h'} & \Sigma A \end{array}$$

where f', g', h' represent f, g, h, respectively, and both squares are cocartesian.

Theorem. *Let $\mathscr{C}$ be a stable ∞-category. Then $\mathrm{h}\mathscr{C}$ with the additional structure specified above is a triangulated category.*

See [HA, 1.1.2] for the proof. We will only explain why the morphism that one gets in (TR3) fails to be unique. Consider distinguished triangles in $\mathrm{h}\mathscr{C}$

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow TA$$

$$A' \xrightarrow{f'} B' \longrightarrow C' \longrightarrow TA'$$

and a commutative diagram in $\mathrm{h}\mathscr{C}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B'. \end{array}$$

The commutativity of the last diagram means that there is an (invertible) 2-morphism in $\mathscr{C}$ which lifts the diagram to a commutative square in $\mathscr{C}$. For each such 2-morphism, we get a morphism $C \to C'$ in $\mathscr{C}$ from the universal property of the cofiber $C = \mathrm{cofib}(f)$ – this induced morphism is essentially unique. However, the choice of a 2-morphism is not unique in general, and there can be different (=non-equivalent) such 2-morphisms that provide (non-homotopic) homotopies for the commutative square in $\mathrm{h}\mathscr{C}$. These different 2-morphisms will generally produce different morphisms $C \to C'$.

Example: Let $\mathscr{C}$ be a stable ∞-category and let

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

be the canonical pushout diagram. Then the induced morphism between the cofibers is $(\Sigma X \xrightarrow{\mathrm{id}} \Sigma X)$. However, if we equip the associated square in $\mathrm{h}\mathscr{C}$ with a different homotopy (= 2-morphism), we will obtain a different morphism between the cofibers in general. For example, if we consider the trivial homotopy:

$$\Delta^1 \times \Delta^1 \xrightarrow{(i,j)\mapsto i+j} \Delta^2 \xrightarrow{(X \to 0 \to \Sigma X)} \mathscr{C},$$

then the induced morphism between the cofibers is the zero morphism.

3. Properties of stable ∞-categories

Proposition. *Let $\mathscr{C}$ be a stable ∞-category, and K a simplicial set. Then* $\mathrm{Fun}(K, \mathscr{C})$ *is again stable.*

Proposition. *Let $\mathscr{C}$ be a pointed ∞-category. Then $\mathscr{C}$ is stable if and only if, $\mathscr{C}$ has finite limits and colimits, and every square in $\mathscr{C}$ is a pushout if and only if it is a pullback.*

Proof. See [HA, Proposition 1.1.3.4]. The sufficiency of these conditions is obvious. For the converse, note that $\mathscr{C}$ has finite (co)products – an argument was sketched above – and pushouts (pullbacks) – because these can be expressed as (co)equalizers, and therefore, as (co)fibers. Suppose that the right square of the diagram

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ 0 & \longrightarrow & X' & \longrightarrow & Y' \end{array}$$

is a pullback and let F be the fiber of f, i.e., the left square is a (co)fiber sequence. Then the composite square is again a fiber sequence. Since $\mathscr{C}$ is stable, the composite diagram is also a cofiber sequence, which then implies that the right square is also a pushout. The converse is similar. □

Proposition. *Let $F : \mathscr{C} \to \mathscr{C}'$ be a functor between stable ∞-categories. Then the following are equivalent:*
(1) F preserves the zero object and fiber sequences.
(2) F is left exact, that is, F preserves all finite limits.
(3) F is right exact, that is, F preserves all finite colimits.

Proof. See [HA, Proposition 1.1.4.1]. The proof is based on the arguments of the last proposition. □

We say that F *exact* if it satisfies these equivalent properties. Note that an exact functor $F \colon \mathscr{C} \to \mathscr{C}'$ induces an exact functor $\mathrm{h}F \colon \mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{C}'$ between triangulated categories.

References

[HA] Jacob Lurie, *Higher algebra*. Available online: `https://www.math.ias.edu/~lurie/`